

Streamline Tracing on General Triangular or Quadrilateral Grids

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Summary

Streamline methods have received renewed interest over the past decade as an attractive alternative to traditional finite-difference (FD) simulation. They have been applied successfully to a wide range of problems including production optimization, history matching, and upscaling. Streamline methods are also being extended to provide an efficient and accurate tool for compositional reservoir simulation. One of the key components in a streamline method is the streamline tracing algorithm. Traditionally, streamlines have been traced on regular Cartesian grids using Pollock's method. Several extensions to distorted or unstructured rectangular, triangular, and polygonal grids have been proposed. All of these formulations are, however, low-order schemes.

Here, we propose a unified formulation for high-order streamline tracing on unstructured quadrilateral and triangular grids, based on the use of the stream function. Starting from the theory of mixed finite-element methods (FEMs), we identify several classes of velocity spaces that induce a stream function and are therefore suitable for streamline tracing. In doing so, we provide a theoretical justification for the low-order methods currently in use, and we show how to extend them to achieve high-order accuracy. Consequently, our streamline tracing algorithm is semi-analytical: within each gridblock, the streamline is traced exactly. We give a detailed description of the implementation of the algorithm, and we provide a comparison of low- and high-order tracing methods by means of representative numerical simulations on 2D heterogeneous media.

Introduction

Streamline simulation is now accepted as a practical tool for reservoir simulation. It represents a fast alternative to the classical FD or finite-volume (FV) methods. However, streamline simulation is still a young technology and does not offer the same capabilities as more traditional methods. Here, we investigate the extension of the streamline method to simulate problems on unstructured or highly distorted grids with full tensor permeability fields.

In streamline simulation, the flow problem (pressure equation) and the transport problem (saturation equations) are solved sequentially in an operator-splitting fashion. The transport problem is solved along the streamlines using a 1D formulation of the transport equation expressed in terms of the time-of-flight variable (Bradvedt et al. 1993; Batycky et al. 1997; King and Datta-Gupta 1998). A background simulation grid is used to solve the flow problem and trace the streamlines. Therefore, extension of the streamline method to general triangular or quadrilateral grids hinges on the ability to: (1) properly discretize the pressure equation, and (2) accurately trace the streamlines on these advanced grids.

These two problems are linked. The key link between discretization and streamline tracing resides in the velocity field description. To each discretization corresponds a particular form of velocity field, and the streamline tracing algorithm has to be adapted to each type of velocity field.

Pollock (1988) derived a streamline tracing method based on a particle tracking concept designed for an FD method on Cartesian quadrilateral grids. The FD method is conservative at the element

level: the elements are mass-balance control volumes and therefore, the fluxes at the faces of the elements are continuous. It is possible to use these fluxes to reconstruct the velocity field inside each element and then integrate the streamline. Pollock's algorithm is semianalytical: given the interpolated velocity field inside each element, the streamline is traced exactly.

Cordes and Kinzelbach (1992) studied the problem of streamline tracing for FEMs. In FEMs, the pressure nodes are located at the vertices of the simulation grid, also called the primal grid. Mass balance is enforced around each node using a control volume construction. The pressure nodes lie at the center of these polygonal control volumes that form a dual grid. The velocity field provided by the FEM on the primal grid is then post-processed to obtain continuous fluxes at the faces of the control volumes. The control volumes are then decomposed into triangles, on which the streamlines are traced.

Prévost et al. (2001) proposed another flux recovery technique, decomposing the control volumes of the FEM discretization into quadrilaterals. Pollock's algorithm, extended to distorted quadrilaterals, is then used to trace streamlines on each quadrilateral composing the control volume. Prévost et al. also extended the method for use with multipoint flux approximation (MPFA) (Aavatsmark 2002), an extension of the classical two-point flux FV method used in the reservoir simulation community.

As noticed by Hægland (2003), Matringe and Gerritsen (2004), and Jimenez et al. (2005), these tracing algorithms can yield inaccurate or unphysical results in terms of streamline location, time-of-flight, and/or arc-length.

The objective of this paper is to develop an accurate streamline tracing method for mixed finite-element and multipoint-flux approximations. To do so, we first develop the method using the mathematical framework of the mixed FEM (MFEM). We then extend the algorithm to MPFA, exploiting the link between both discretization methods. Our streamline tracing method does not use any flux-recovery technique: the fluxes obtained from MFEM and MPFA satisfy a discrete mass-balance condition at the element level. The algorithm can handle unstructured grids formed by general triangles or quadrilaterals and offers two possible orders of accuracy. The lowest-order accurate version of our algorithm corresponds to the commonly used streamline-tracing methods on these grids: the streamline is a straight line on a triangular element and reduces to Pollock's method on Cartesian quadrilateral grids. The algorithm is, for now, limited to 2D grids, but its extension to three dimensions is discussed.

In this paper, we start by recalling the main features of MFEM. We then introduce the mathematical framework necessary to understand streamline tracing issues including the functional spaces defining the velocity fields. In view of their specific properties, these spaces are particularly well suited for streamline tracing. In a second part, the streamline tracing algorithm itself is proposed, and its implementation for both triangular and quadrilateral grids is described in detail. Next, we extend the method to MPFA discretizations. We subsequently show results of our tracing algorithm on triangular and quadrilateral grids populated with heterogeneous permeability fields. Finally, we give some conclusions and describe ongoing work and future directions.

MFE Approximation

Mathematical Model. We use a prototype of the pressure equation in reservoir simulation models. The mathematical model encompasses Darcy's law (Eq. 1) and mass conservation (Eq. 2):

$$\mathbf{K}^{-1}\mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \Omega, \dots\dots\dots (1)$$

$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega. \dots\dots\dots (2)$$

In Eqs. 1 and 2, \mathbf{u} is the Darcy velocity, p is the pressure, \mathbf{K} a full tensor permeability, f is the source term (e.g., injection and production wells), and Ω is the domain of interest. This system of equations must be supplemented with appropriate boundary conditions. For expositional simplicity, we assume no-flow boundary conditions throughout:

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{in } \Gamma$$

where Γ is the boundary of the domain, and \mathbf{n} is the outward unit normal vector. The essence of the MFEM is to solve for pressure and velocity simultaneously. The starting point for the MFEM is the weak form of the problem:

$$\int_{\Omega} \mathbf{v} \cdot \mathbf{K}^{-1}\mathbf{u} \, d\Omega - \int_{\Omega} \nabla \cdot \mathbf{v} p \, d\Omega = 0, \dots\dots\dots (3)$$

$$\int_{\Omega} q \nabla \cdot \mathbf{u} \, d\Omega = \int_{\Omega} q f \, d\Omega. \dots\dots\dots (4)$$

In Eqs. 3 and 4, \mathbf{v} is the velocity test function and q is the pressure test function.

Discretization. To discretize the continuum problem given by Eqs. 3 and 4, two types of shape functions are used: one for the pressure and one for the velocity. It is well known that not all combinations of pressure and velocity interpolation yield a convergent approximation (Brezzi and Fortin 1991). We are interested in numerical approximations in which the pressure unknowns are located at the center of each element and the flux unknowns, used to interpolate the velocity field, are on the element edges. The elements are the control volumes on which the mass-balance condition is enforced. **Fig. 1** represents the locations of the unknowns on the reference elements for the lowest-order approximation used in this paper.

The pressure and velocity fields are interpolated from the pressure and flux unknowns using the shape functions

$$\mathbf{u} = \sum_{i=1}^{N_{\text{edges}}} U_i \mathbf{N}_i^u, \dots\dots\dots (5)$$

$$p = \sum_{j=1}^{N_{\text{elements}}} P_j N_j^p, \dots\dots\dots (6)$$

where \mathbf{u} and p are the global velocity and pressure fields, \mathbf{N}_i^u , N_j^p are the velocity and pressure shape functions, and U_i , P_j are the flux and pressure unknowns.

The discretization of the system yields an indefinite linear system of the form

$$\begin{pmatrix} \mathbf{A} & -\mathbf{B}^t \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \mathbf{R}^u \\ \mathbf{R}^p \end{pmatrix}, \dots\dots\dots (7)$$

where \mathbf{U} and \mathbf{P} are the vectors of unknown fluxes and pressures, \mathbf{R}^u and \mathbf{R}^p are the flux and pressure right-side vectors, \mathbf{A} is a square matrix of size $N_{\text{edges}} \times N_{\text{edges}}$, and \mathbf{B} is a matrix of size

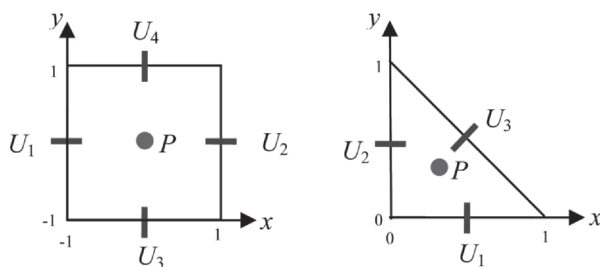


Fig. 1—Location of MFEM unknowns on the reference elements: pressure (P) and fluxes (U_1 , U_2 , U_3 , U_4).

$N_{\text{elements}} \times N_{\text{edges}}$. Because of the indefinite character of the system, an augmented Lagrangian method known as Uzawa's algorithm (Fortin and Glowinski 1983) is used to solve the problem.

The MFEM just described is mass conservative at the element level: the mass-balance condition is enforced on each control volume on which the rock properties are defined. Therefore, in the absence of sources and sinks, the MFEM fluxes yield a divergence-free velocity field; that is, the sum of the fluxes over all the element faces is identically zero.

Velocity Spaces. The global velocity field belongs to the space (Brezzi and Fortin 1991)

$$H(\text{div}, \Omega) = \{\mathbf{u} | \mathbf{u} \in (L^2(\Omega))^2; \nabla \cdot \mathbf{u} \in L^2(\Omega)\}, \dots\dots\dots (8)$$

where $L^2(\Omega)$ is the space of square integrable functions on Ω . The functional space $H(\text{div}, \Omega)$ is designed so that the normal component of the velocity field exists on the boundary of the domain. The integral of the normal trace of the velocity field along a boundary is precisely the volumetric flux across this boundary. We thus understand the importance of being able to construct a well-defined normal trace of the velocity field.

We employ a conforming approximation; that is, we look for a discrete velocity field in a finite-dimensional subspace of the infinite-dimensional space $H(\text{div}, \Omega)$. To force the global velocity field to belong to $H(\text{div}, \Omega)$, the discrete approximation must satisfy two conditions (Brezzi and Fortin 1991): (1) the velocity field must belong to $H(\text{div}, K)$ locally on each element K of Ω ; and (2) the trace of the normal component of the velocity must be continuous between adjacent elements.

The Space RT_0^0 . The simplest polynomial subspace conforming in $H(\text{div}, \Omega)$ is the lowest-order Raviart-Thomas (1977) space, $RT_0(K)$. We are interested in the restriction of $RT_0(K)$ to functions of zero divergence:

$$RT_0^0(K) = \{\mathbf{u} | \mathbf{u} \in RT_0(K); \nabla \cdot \mathbf{u} = 0\}. \dots\dots\dots (9)$$

Velocity fields in $RT_0(K)$ are described by a constant trace along element edges, as shown in **Fig. 2**. Therefore, knowledge of the fluxes across each of the edges of an element is sufficient to fully describe an RT_0 velocity field. Thus, three degrees of freedom are needed to describe RT_0 on triangles and four on quadrilaterals. Because of the divergence-free constraint, the dimension of RT_0^0 is further reduced by one:

$$\dim(RT_0^0(K)) = 3 - 1 = 2 \quad \text{for a triangular element,}$$

$$\dim(RT_0^0(K)) = 4 - 1 = 3 \quad \text{for a quadrilateral element.}$$

The Space BDM_1^0 . A higher-order description of the velocity field can be obtained by using a linear velocity profile along element edges, as shown in **Fig. 3**.

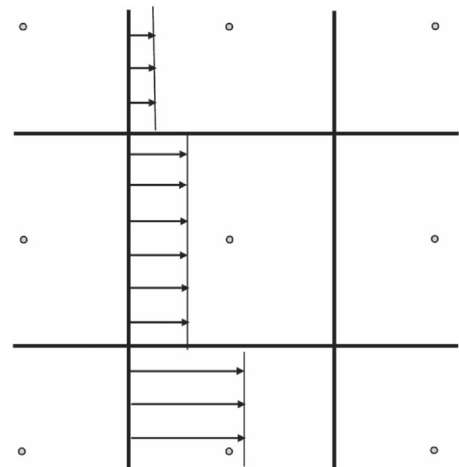


Fig. 2—Velocity profile across element edges in an RT_0 velocity field (constant trace of the velocity along the edges).

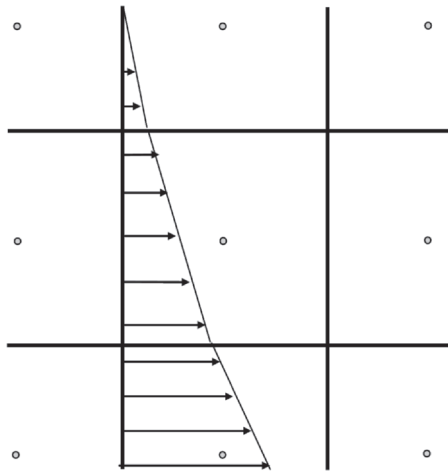


Fig. 3—Velocity profile across element edges in a BDM₁ velocity field (linear variation of the trace of the velocity along the edges).

The polynomial space conforming in $H(\text{div}, \Omega)$ that corresponds to a velocity field with a linear variation of the trace along the element edges is the Brezzi-Douglas-Marini (1985) space of order 1, $\text{BDM}_1(K)$. Once again, because we are considering incompressible flow, we further restrict this space by enforcing the divergence-free condition on the velocity field. We thus define

$$\text{BDM}_1^0(K) := \{\mathbf{u} | \mathbf{u} \in \text{BDM}_1(K); \nabla \cdot \mathbf{u} = 0\}, \dots \dots \dots (10)$$

the restriction of $\text{BDM}_1(K)$ to fields of zero divergence. To fully describe this space, two unknowns per edge are necessary and the divergence free condition reduces the dimension by one unknown, so

$$\dim(\text{BDM}_1^0(K)) = 5 \quad \text{for a triangular element,}$$

$$\dim(\text{BDM}_1^0(K)) = 7 \quad \text{for a quadrilateral element.}$$

The order of accuracy of the velocity field has to be compatible with the order of accuracy of the pressure field in order to obtain a convergent scheme. The numerical approximation must satisfy the Babuška-Brezzi condition (Brezzi and Fortin 1991). Because of our piecewise constant pressure description, the order of accuracy of the velocity field is limited. To our knowledge, there is no polynomial space more accurate than $\text{BDM}_1^0(K)$ and conforming in $H(\text{div}, \Omega)$ that is compatible with a piecewise constant pressure field.

Existence of a Stream Function. The stream function Ψ is the imaginary part of the complex potential. For 2D problems, the stream function can be linked to the velocity field through

$$u_x = \frac{\partial \Psi}{\partial y}, \dots \dots \dots (11)$$

$$u_y = -\frac{\partial \Psi}{\partial x}, \dots \dots \dots (12)$$

where u_x and u_y are the x - and y -components of the velocity. The stream function has the property of being constant along a streamline—a property we exploit in the streamline tracing algorithm described in the next section.

A large class of mixed finite-element spaces that induce a stream function has been identified (Juanes and Matringe 2007). The RT_0^0 and BDM_1^0 spaces belong to this class and are therefore particularly well suited for streamline tracing.

Streamline Tracing Method

Most streamline tracing algorithms are based on particle-tracking concepts. The streamline is traced by following a particle of fluid in time. Because of the discretization of the domain into elements, the velocity field is defined elementwise and not globally. It is therefore natural to trace a streamline by segments, each segment corresponding to an underlying element.

Our algorithm uses this concept. The streamline tracing procedure can be summarized as follows:

- Start at a “launching” point in the domain. The launching point defines in space and time the fluid particle that will be followed.
- Find the element this launching point belongs to.
- Trace the streamline downstream toward a sink (producing well). The fluid particle is followed forward in time. We visit sequentially all elements crossed by the streamline:
 - Trace the streamline downstream in the current element from the entry point.
 - Store the exit point of the streamline.
 - Move to the next downstream element.
 - Continue until a sink is reached.
- Trace the streamline toward a source (injection well). The fluid particle is followed backward in time.
 - Trace the streamline upstream in the current element from the entry point.
 - Store the exit point.
 - Move to the next upstream element.
 - Continue until a source is reached.

The streamlines are stored only by placing in memory the points of intersection of the streamline with the simulation grid. The time-of-flight and arc-length of the streamline in each element are also stored. In the streamline method, these variables are used in the solution of the transport problem.

Streamline Tracing Within an Element. The differences between most streamline tracing algorithms lie in the way the streamlines are traced within each element. Some methods, like Pollock’s, use an interpolation of the velocity field from the fluxes at the edges of the element. The exit point is then obtained analytically. Alternatively, one can use a numerical timestepping technique (Runge-Kutta or similar) to follow the particle in time within each element.

We take a different approach. We exploit the fact that the velocity fields from the MFEM induce a stream function. Because the stream function is constant along a streamline, the equation defining a streamline that passes through a point (x_o, y_o) is

$$\Psi_K(x, y) = \Psi_K(x_o, y_o), \dots \dots \dots (13)$$

where Ψ_K is the stream function in an element K . As a result, the problem of tracing a streamline is simplified from the solution of an ordinary differential equation to an algebraic equation. For practical purposes, only the exit point at a given element is needed. Because the stream function has an analytical expression, an efficient Newton method can be used to solve the algebraic equation to machine precision. Therefore, the tracing algorithm proposed here provides an exact streamline or, more precisely, exact streamline locations at the interfaces between elements of the simulation grid.

The actual tracing is performed on the reference element (see Fig. 1). This is done for two reasons. First, the analytical expressions of the velocity field and the stream function are known on reference space rather than on physical space. Second, the location of the edges of the element is fixed and independent of the actual position and distortion of the element on the simulation grid.

The procedure is as follows: given an entry point on the reference element, the exit point is computed on the reference element by solving Eq. 13. This location will be used as the entry point for the adjacent element. It can be mapped onto physical space by a linear transformation (for triangular elements) or a bilinear transformation (for quadrilateral elements). Inverting the mapping for quadrilateral grids is more involved (Haegland 2003), but this is required at most once for each streamline, only to define the starting point if the launching location is defined in physical space.

Depending on the functional form of the stream function, Eq. 13 may lead to multiple solutions on the boundary of a given element. It can be shown that this situation is never encountered for stream functions derived from RT_0^0 velocity fields, but it does indeed occur for BDM_1^0 velocity fields. However, a simple logic based on comparing the direction of the velocity at each of the

potential solutions with the direction at the entry point allows one to identify the exit point uniquely.

Computation of the Time-of-Flight. Mapping the elements from physical space to a reference element and the use of the stream function allows one to efficiently compute the streamline and, in particular, the exact exit point. However, one crucial ingredient of streamline simulation is the accurate calculation of the time-of-flight along a streamline:

$$\tau = \int_{SL} \frac{1}{|\mathbf{u}(s)|} ds \quad (14)$$

where s represents the arc-length along the streamline SL . For consistency with the rest of the streamline tracing framework, one must be able to evaluate the integral in Eq. 14 in reference space. The correct transformation of velocity from the reference space ($\hat{\mathbf{x}}$) to the physical space (\mathbf{x}) is given by the Piola transform (Brezzi and Fortin 1991):

$$\mathbf{u}(\mathbf{x}) = \frac{1}{J} \mathbf{J}(\hat{\mathbf{x}}) \hat{\mathbf{u}}(\hat{\mathbf{x}}), \quad (15)$$

where $\hat{\mathbf{u}}$ is the velocity at point $\hat{\mathbf{x}}$ in reference space and \mathbf{u} is the velocity in physical space at the mapped location \mathbf{x} ; \mathbf{J} is the Jacobian matrix of the transformation from reference to physical space; and J is the determinant of the Jacobian. Simply substituting the velocity \mathbf{u} by its inverse Piola transform $\hat{\mathbf{u}}$ in Eq. 15 yields an incorrect time-of-flight. The exact expression of the time-of-flight as an integral on reference space is given by (Juanes and Matringe 2007):

$$\tau = \int_{SL} \frac{1}{|\hat{\mathbf{u}}(\hat{s})|} J(\hat{s}) d\hat{s}. \quad (16)$$

It is worthwhile noticing at this point that the isoparametric mapping is affine for triangular elements. The Jacobian is therefore constant and can be taken out of the integral. For quadrilateral elements, however, the Jacobian varies inside the element. Prévost et al. (2001) used the value of the Jacobian at the center of the element as an approximation, but Hægland (2003) showed how this choice could lead to erroneous results and recommended keeping the Jacobian in the integral.

The time-of-flight cannot, in general, be integrated analytically and one must resort to numerical quadrature.

Low-Order Tracing. In the previous sections, we described the streamline tracing methodology for general triangular and quadrilateral grids. It is based on the use of the stream function induced by mixed finite-element velocity fields on the reference element. We now give the expressions of the velocity field and the stream function for the lowest-order accurate streamline tracing method, which uses the RT_0^0 space. As explained earlier, RT_0^0 defines velocity fields whose normal component has constant trace along the edges. Only one degree of freedom per edge—the overall flux through the edge—is necessary to define an RT_0^0 velocity field and the corresponding stream function. We also recall that all edge fluxes are not independent because they are subject to the zero divergence condition.

Triangular Elements. The RT_0^0 velocity field on triangular elements has only two degrees of freedom. It is therefore a constant velocity field:

$$\begin{cases} u_x = a_1 \\ u_y = a_2 \end{cases}, \quad (17)$$

where a_1 and a_2 are two constants. Direct integration of the above velocity field yields the stream function:

$$\Psi(x, y) = -a_2 x + a_1 y. \quad (18)$$

Therefore, for a triangular RT_0^0 element, the streamline is a straight line. This corresponds to the classical low-order streamline

tracing method used by many authors (Durlafsky 1991; Cordes and Kinzelback 1992; Mose et al. 1994; Prevost et al. 2001).

Quadrilateral Elements. The RT_0^0 velocity field on quadrilateral elements is

$$\begin{cases} u_x = a_1 + b_1 x \\ u_y = a_2 - b_1 y \end{cases}, \quad (19)$$

This velocity field (Eq. 19) yields the bilinear stream function:

$$\Psi(x, y) = -a_2 x + a_1 y + b_1 xy. \quad (20)$$

For Cartesian quadrilateral grids, the tracing algorithm based on RT_0 mixed finite-element functions reduces to the well-known method proposed by Pollock (1988). For our reference element shown in Fig. 1, Pollock proposed a linear interpolation of the velocity field:

$$\begin{aligned} u_x &= -\frac{U_1}{4}(1-x) + \frac{U_2}{4}(1+x), \\ u_y &= -\frac{U_3}{4}(1-y) + \frac{U_4}{4}(1+y). \end{aligned} \quad (21)$$

Because of the zero divergence constraint,

$$U_1 + U_2 + U_3 + U_4 = 0, \quad (22)$$

one can establish a direct correspondence between Pollock's expressions and the RT_0^0 velocity field:

$$\begin{aligned} a_1 &= \frac{U_2 - U_1}{4} \\ a_2 &= \frac{U_4 - U_3}{4} \\ b_1 &= \frac{U_1 + U_2}{4} = -\frac{U_3 + U_4}{4} \end{aligned} \quad (23)$$

High-Order Tracing. To obtain higher-order accuracy, the velocity is described by a vector field that displays a linearly varying flux across element edges. The description of such velocity fields requires two unknowns per edge: the first provides the flux through the face, and the second indicates the tilt of the velocity profile around the mean velocity. This is accomplished by the use of functions belonging to BDM_1^0 .

Triangular Elements. The BDM_1^0 velocity field on triangular elements is of the form

$$\begin{cases} u_x = a_1 + b_1 x + c_1 y \\ u_y = a_2 + b_2 x - b_1 y \end{cases}, \quad (24)$$

where a_1 , a_2 , b_1 , b_2 , and c_1 are constants. An analytical expression of the stream function can be derived from the velocity field in Eq. 24 (Juanes and Matringe 2007):

$$\Psi(x, y) = -a_2 x + a_1 y + b_1 xy - \frac{b_2}{2} x^2 + \frac{c_1}{2} y^2. \quad (25)$$

Quadrilateral Elements. On quadrilateral elements, BDM_1^0 is the space of velocity fields of the form

$$\begin{cases} u_x = a_1 + b_1 x + c_1 y - rx^2 - 2sxy \\ u_y = a_2 + b_2 x - b_1 y + 2rxy + sy^2 \end{cases}, \quad (26)$$

where a_1 , a_2 , b_1 , b_2 , c_1 , s , and r are constants. Integrating the velocity field yields (Juanes and Matringe 2007):

$$\Psi(x, y) = -a_2 x + a_1 y + b_1 xy - \frac{b_2}{2} x^2 + \frac{c_1}{2} y^2 - rx^2 y - sxy^2. \quad (27)$$

Tracing Streamlines From an MPFA Solution

So far, the streamline tracing algorithm relies entirely on a mixed finite-element discretization of the pressure equation. We have seen, however, that for low-order tracing on Cartesian grids, the proposed method is identical to Pollock's method, which was de-

signed for finite-difference schemes. In this section, we show that our streamline tracing framework can be applied to more general finite-volume discretizations. This opens the possibility for high-order streamline tracing on solutions to the pressure equation obtained with conventional reservoir simulators.

MPFA methods were introduced in reservoir simulation to handle problems described with full tensor permeabilities on general polygonal grids. The method takes its name from the stencil used to evaluate the flux through the face of an element. The simplest FV method, the two-point flux approximation, only considers the pressure at two adjacent elements for the computation of the flux through an edge.

In MPFA, the flux through an edge is computed using an extended stencil, and makes use of the pressures for all the elements in contact with the edge. This extended stencil allows for a more accurate computation of the flux when the grid is distorted and permits the use of full tensor permeabilities. Although the precise way in which these fluxes are computed is unimportant for our purpose, it is important to note that MPFA computes two subfluxes per edge. These subfluxes correspond to the fluxes through each half-edge. The total flux through the edge is the sum of these two subfluxes.

To use our streamline tracing algorithm, we have to relate the MPFA quantities to the degrees of freedom of the mixed finite-element formulation.

Low-Order Tracing. When using RT_0 mixed finite elements, the degrees of freedom correspond to the fluxes through the element edges. Therefore, the MPFA total edge fluxes can be used directly in our streamline tracing method as if they were RT_0^0 -MFEM fluxes.

High-Order Tracing. When using BDM_1 mixed finite elements, there are two degrees of freedom per edge: the first corresponds to the total flux through the edge, and the second to a flux recirculation responsible for the linear variation in the normal component of the velocity along the edge. In an MPFA formulation, there are also two degrees of freedom per edge, corresponding to the two subfluxes. **Fig. 4** illustrates what we are trying to achieve: recovering a BDM_1^0 -MFEM type linear velocity profile from the two MPFA subfluxes. The MPFA subfluxes are represented by the solid line, the RT_0^0 -MFEM profile is in dotted-dashed, and the BDM_1^0 -MFEM profile is the dashed line. Let us denote the degrees of freedom in the BDM_1 MFEM by U and V , and the two subfluxes in the MPFA formulation by f_1 and f_2 . It is a simple

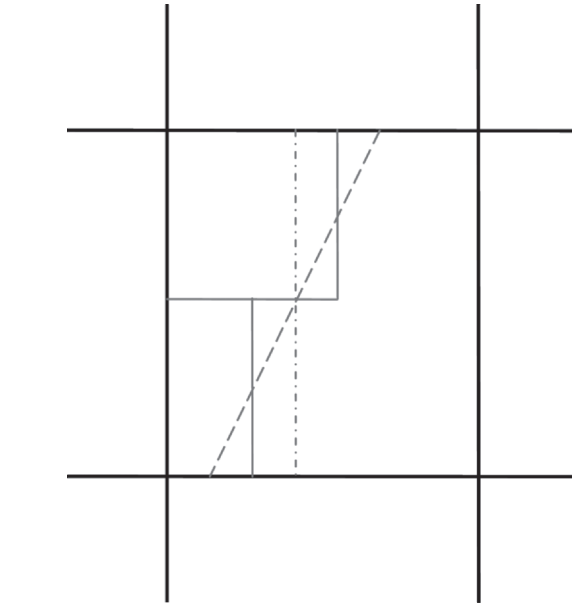


Fig. 4—Velocity profiles from MPFA (thin red line), RT_0^0 -MFEM (dotted-dashed red line), and BDM_1^0 -MFEM (dashed red line).

exercise to show (rigorously) that the BDM_1 degrees of freedom are related to the MPFA subfluxes as follows:

$$\begin{aligned} U &= f_1 + f_2 \\ V &= \frac{1}{4} (f_1 - f_2) \end{aligned} \quad \dots\dots\dots (28)$$

We conclude that the proposed high-order tracing algorithm can make direct use of an MPFA finite-volume solution to the pressure equation.

Results

We test our streamline tracing algorithm using MFEM with RT_0 and BDM_1 spaces for the solution of the flow problem. The streamlines are traced with the order of accuracy corresponding to the discretization.

Quarter Five-Spot Model With Distorted Grids. We test and compare the performance of low-order and high-order tracing on a test case with isotropic, homogeneous permeability. The domain corresponds to a quarter of a five-spot pattern with the injector at the bottom left corner and the producer at the top right corner.

In **Fig. 5**, we show the streamlines traced with the RT_0 (low-

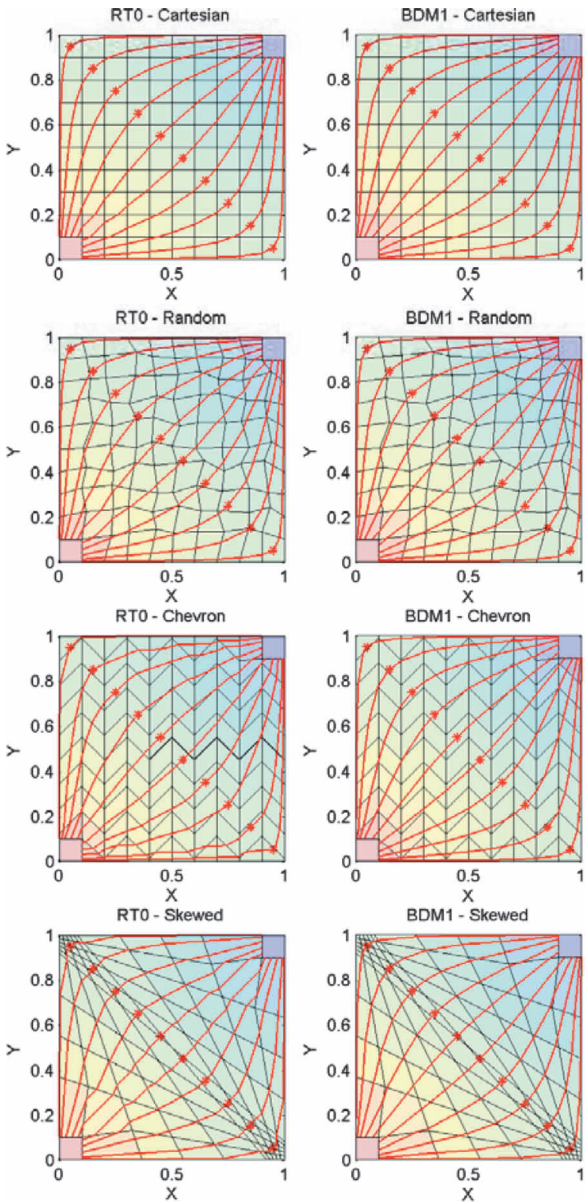


Fig. 5— RT_0^0 and BDM_1^0 streamlines on quadrilateral grids.

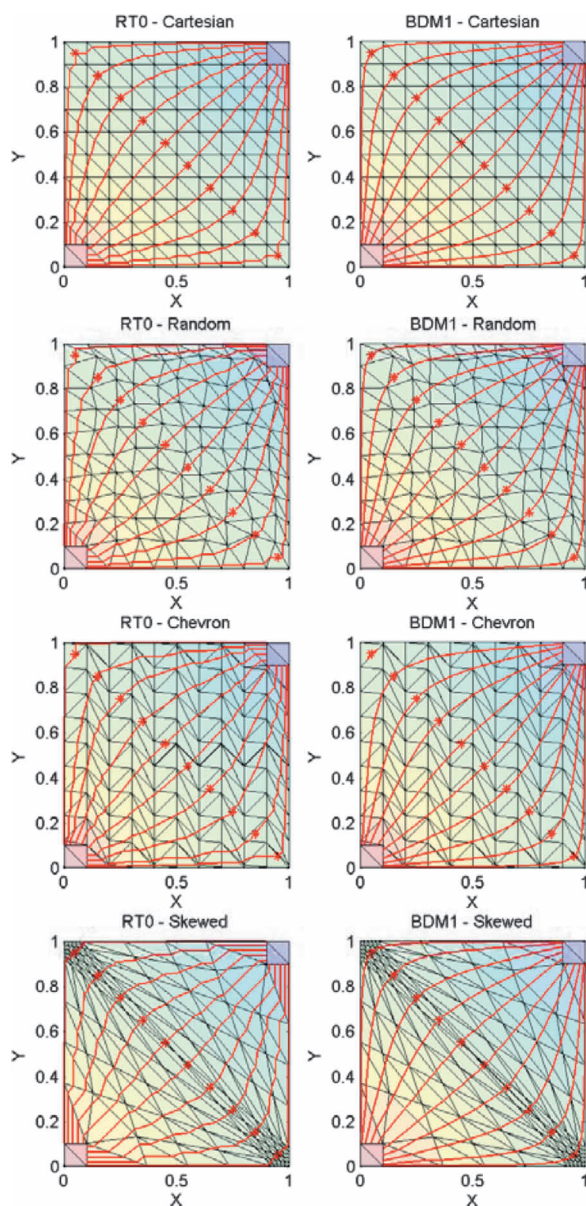


Fig. 6— RT_0^0 and BDM_1^0 streamlines on triangular grids.

order) and BDM_1 (high-order) methods on four different quadrilateral grids. The reference grid is a regular, Cartesian grid. The second type of grid used is obtained by randomly moving the grid nodes around their initial location on a Cartesian grid. The third type is a chevron grid, with inclined edges forming an angle of 40° with the vertical edges. The fourth type of grid is a highly skewed grid that leads to very acute angles for some elements. The streamlines, shown in red, are launched from points located on the diagonal of the field and equally spaced.

The triangular grids in Fig. 6 are obtained by splitting in half the quadrilaterals of the previous grids. The quadrilateral grids contain 100 elements and the triangular grids contain 200 elements.

The main observation is that the high-order discretization and tracing method based on BDM_1 mixed finite elements produce much better streamlines. While RT_0 elements yield jagged and heavily grid-dependent streamlines, BDM_1 elements result in smooth, physical streamlines which show very little sensitivity to the simulation grid. This is, of course, a combined effect of higher-order accuracy in the discretization and the streamline tracing.

Heterogeneous System With a Fully Unstructured Grid. As an illustration of our streamline tracing capabilities, we show in Fig. 7 the results of mixed finite-element simulations and stream-

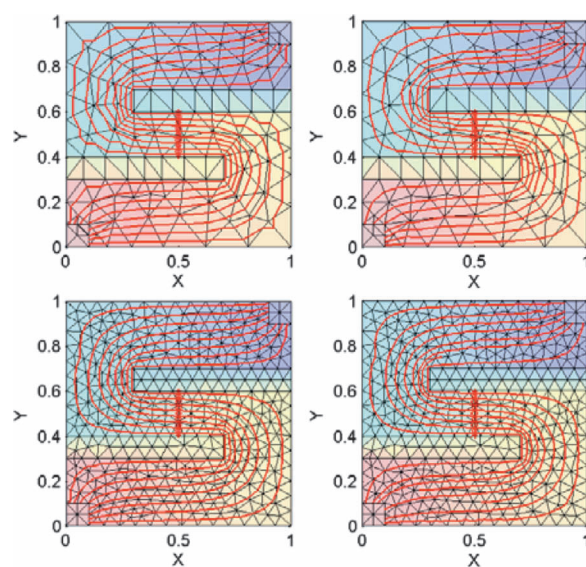


Fig. 7— RT_0^0 (left column) and BDM_1^0 (right column) streamlines on unstructured triangular grids with flow barriers.

line tracing on fully unstructured triangular meshes. The medium contains two low-permeability flow barriers. All boundaries are no-flow boundaries. An injector is located at the bottom left corner and a producer at the top right corner of the domain. The streamlines are traced from the center of the field. The launching locations are located on a vertical line between the two flow barriers. Two grids of different resolution were used, to investigate the behavior of the low-order and high-order methods as the grid is refined.

Once again, the BDM_1^0 results show smoother streamlines that are less dependent on the grid, compared with those obtained with RT_0^0 .

Conclusions

In this paper, we presented a unified framework for streamline tracing on general triangular and quadrilateral grids. The method is based on the use of the stream function to define the streamline location. Given a velocity interpolation, the streamline is traced exactly. One of the key steps is the identification of appropriate velocity spaces that induce a stream function. Such velocity spaces (RT_0 and BDM_1) are taken from the theory of MFEMs (Juanes and Matringe 2007). In this way, we justify theoretically low-order tracing algorithms currently in use (Pollock's method), and we show how to choose the velocity interpolation for higher-order tracing.

Our numerical simulations illustrate the improvement in the streamlines obtained with the high-order, BDM_1 -based method in comparison with the low-order, RT_0 -based method.

In principle, the tracing algorithm relies on a mixed finite-element solution of the pressure equation. However, it is important to note that the streamline tracing framework can be used without modification in conjunction with finite-volume solutions of the pressure equation that use either a two-point flux (low-order tracing) or a multipoint flux approximation (high-order tracing). A comparison of low-order and high-order tracing in the context of multipoint flux approximation schemes is currently being addressed.

Conceptually, the framework can be extended to 3D problems, but the extension of the streamline tracing method itself requires the derivation of dual stream functions for complicated velocity fields. These derivations can be mathematically involved, and thus the streamline tracing algorithm is not yet developed for 3D grids.

Nomenclature

- \mathbf{A} , \mathbf{B} = matrices in linear system of the MFEM problem
- f = source term
- J = determinant of the Jacobian of the isoparametric transformation

\mathbf{J} = Jacobian matrix of the isoparametric transformation
 \mathbf{K} = permeability tensor
 \mathbf{n} = outward unit normal vector
 N_{edges} = number of edges
 N_{elements} = number of elements
 \mathbf{N}_j^p = pressure shape function of element j
 \mathbf{N}_i^u = velocity shape function of edge i
 p = pressure
 P_j = pressure at element j
 \mathbf{P} = vector of unknown pressures
 q = pressure test function
 $\mathbf{R}^p, \mathbf{R}^u$ = pressure and flux right-side vectors
 s = arc length
 \mathbf{u} = Darcy velocity
 u_x, u_y = x - and y -components of the velocity field
 U_i = flux through edge i
 \mathbf{U} = vector of unknown fluxes
 \mathbf{v} = velocity test function
 Γ = boundary of the domain of simulation
 τ = time of flight
 Ψ = stream function
 Ω = domain of simulation

Acknowledgments

The authors gratefully acknowledge financial support from the members of the affiliate program of the Stanford University Petroleum Research Institute for Reservoir Simulation (SUPRI-B).

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