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Determination of the Wave Structure of the Three-Phase Flow Riemann Problem

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Abstract. In a previous paper (*Transp. Porous Media*, **55**(1): 47–70), algorithms are given for computing the analytical solution to the three-phase Riemann problem. Application of those algorithms requires that the wave configuration is known. The purpose of this note is to provide a procedure to determine the wave structure for any initial and injected saturation states.

Key words: three-phase flow, Riemann problem, wave structure, hyperbolic, shock.

1. Wave Structure Algorithm

The solution to the Riemann problem of three-phase flow in porous media is discussed in Juanes and Patzek (2004). It is found that, when the relative permeability functions satisfy certain physically-based conditions, then: (1) the first-order system of saturation equations is strictly hyperbolic; and (2) both characteristic fields are nongenuinely nonlinear, with single, connected inflection loci. It is concluded that there are nine admissible solution types, and efficient algorithms are given for the implementation of the analytical solution in each of these cases. The paper gives admissibility criteria for each solution type. Therefore, the validity of the assumed wave structure can be checked once the solution has been computed. The paper leaves open the question of how the solution *type* is obtained. We address this question here, and we give an algorithm for the actual determination of the structure of the solution. The algorithm is designed to converge for all initial and injected states, and we show that the correct solution structure is usually obtained after one iteration.

1.1. TWO-PHASE FLOW

We illustrate the algorithm for determining the wave structure with the much simpler two-phase flow case. The problem is described by a scalar

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hyperbolic equation,

$$\partial_t u + \partial_x f = 0, \tag{1}$$

where u is the water saturation, and $f(u)$ is the fractional flow function. A standard feature of the two-phase flow model is that the flux function f is S-shaped: it has a unique inflection point at u_0, which corresponds to a maximum value of the derivative f'. In this case, it is well-known that the solution to the Riemann problem with left state u_l and right state u_r may only involve a rarefaction, a shock, or a composite rarefaction-shock.

A *single rarefaction* is admissible only if the left and right states are on the same convexity region, with the right state closer to the inflection point: $u_0 \leqslant u_r < u_l$, or $u_l < u_r \leqslant u_0$. A *single shock* is admissible if it satisfies the Lax entropy criterion: $f'(u_l) > \sigma > f'(u_r)$, where $\sigma = (f(u_l) - f(u_r))/(u_l - u_r)$ is the shock speed. A *composite wave* is admissible if the rarefaction and the shock are both admissible individually. A necessary (but not sufficient) condition for the solution to be a rarefaction-shock is that the left and right state lie on opposite sides of the inflection point, so that the characteristic speed is not monotonic: $u_l > u_0 > u_r$, or $u_l < u_0 < u_r$. In Figure 1, we present an algorithm for obtaining the wave structure in two-phase flow.

- Given left and right states: u_l, u_r
- Trial shock speed: $\sigma^{\text{trial}} = \dfrac{f(u_l) - f(u_r)}{u_l - u_r}$

  IF $f'(u_l) > \sigma^{\text{trial}} > f'(u_r)$ THEN

  - $\mathcal{S}$: Single shock with speed $\sigma = \sigma^{\text{trial}}$

  ELSE

    IF $f'(u_l) < f'(u_r)$ & $f''(u_l)f''(u_r) > 0$ THEN
    - $\mathcal{R}$: Single rarefaction

    ELSE
    - $\mathcal{RS}$: Composite rarefaction-shock
      - · Post-shock value u_* such that $f'(u_*) = \sigma_*$
      - · Shock speed: $\sigma_* = \dfrac{f(u_*) - f(u_r)}{u_* - u_r}$

    END

  END

Figure 1. Algorithm for obtaining the wave structure in two-phase flow.

1.2. THREE-PHASE FLOW

Under common assumptions, three-phase flow is described by the system of equations

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{f} = \boldsymbol{0}, \tag{2}$$

where $\boldsymbol{u}$ is the vector of water and gas saturations, and $\boldsymbol{f}$ is the vector of fractional flow functions. We assume that the system is strictly hyperbolic, and that the inflection locus of each characteristic family is a single connected curve, which corresponds to maxima of eigenvalues (Juanes and Patzek 2004, Fig. 2). Under these conditions, the solution to the Riemann problem is a sequence of two waves, each of which may only be a rarefaction, a shock, or a rarefaction-shock.

The *inflection locus* of the i-family is the set of states at which the i-characteristic velocity attains a maximum or a minimum value when moving along integral curves of the i-family. We define, for any saturation state $\boldsymbol{u}$, the quantity

$$V_i(\boldsymbol{u}) := \nabla \nu_i(\boldsymbol{u}) \cdot \boldsymbol{r}_i(\boldsymbol{u}), \tag{3}$$

where ν_i is the i-eigenvalue and $\boldsymbol{r}_i$ is the i-eigenvector of the Jacobian matrix $\boldsymbol{f}'(\boldsymbol{u})$. With this definition, the i-inflection locus is nothing but the contour $V_i = 0$, which separates convexity regions.

The admissibility criterion for the solution is that each wave must be admissible, that is, i-rarefactions must connect saturation states in the same convexity region (with respect to the i-inflection locus); shocks must satisfy the Lax entropy criterion; and rarefaction-shocks of the i-family must connect states on opposite sides of the i-inflection locus. In addition, one must check that the computed intermediate state is inside the saturation triangle and that the two waves are strictly separated.

We compute a trial solution by assuming that both waves are genuine shocks (incidentally, this is the fastest solution type to compute). Once a trial solution has been computed, we can ascertain what the wave structure of the solution would be, *if* the intermediate state were the trial one. This is done for each wave individually using the same arguments as for the two-phase case: a valid i-shock must satisfy the Lax entropy criterion; if the shock is not admissible, and the constant states joined by the i-wave are on the same convexity region, the i-wave is a rarefaction; otherwise it is a rarefaction-shock. Although integral curves and Hugoniot loci do not coincide, they have similar paths. This means that the intermediate state (defined by the intersection of the 1- and 2-waves) is not very sensitive to the solution type, and the procedure usually converges after one iteration. Obviously, if both shocks are admissible, the trial solution is the actual solution and no

1. Given left and right states: $\boldsymbol{u}_l \equiv \boldsymbol{u}_0^{\text{trial}}$, $\boldsymbol{u}_r \equiv \boldsymbol{u}_2^{\text{trial}}$
2. Trial solution: $\mathcal{W}_1\mathcal{W}_2^{\text{trial}} = \mathcal{S}_1\mathcal{S}_2$, $(\boldsymbol{u}_m^{\text{trial}} \equiv \boldsymbol{u}_1^{\text{trial}})$
3. Solve $\mathcal{W}_1\mathcal{W}_2^{\text{trial}}$ configuration, and update wave structure:

   FOR $i = 1, 2$

       IF $\mathcal{W}_i^{\text{trial}} = \mathcal{S}_i$ AND $\nu_i(\boldsymbol{u}_{i-1}^{\text{trial}}) > \sigma_i^{\text{trial}} > \nu_i(\boldsymbol{u}_i^{\text{trial}})$ THEN

       • $\mathcal{W}_i = \mathcal{S}_i$: i-wave is a genuine shock

       ELSE

           IF $V_i(\boldsymbol{u}_{i-1}^{\text{trial}})V_i(\boldsymbol{u}_i^{\text{trial}}) > 0$ THEN

           • $\mathcal{W}_i = \mathcal{R}_i$: i-wave is a rarefaction

           ELSE

           • $\mathcal{W}_i = \mathcal{R}_i\mathcal{S}_i$: i-wave is a rarefaction-shock

           END IF

       END IF

   END FOR
4. Check convergence:

       IF $\boldsymbol{u}_m^{\text{trial}} \notin \triangle$ OR $\sigma_1^{\text{trial}} > \sigma_2^{\text{trial}}$ Set new $\boldsymbol{u}_m^{\text{trial}}$ and GOTO 2

       ELSEIF $\mathcal{W}_1\mathcal{W}_2 = \mathcal{W}_1\mathcal{W}_2^{\text{trial}}$ STOP

       ELSE Set $\mathcal{W}_1\mathcal{W}_2^{\text{trial}} \leftarrow \mathcal{W}_1\mathcal{W}_2$ and GOTO 3

Figure 2. Algorithm for obtaining the wave structure in three-phase flow.

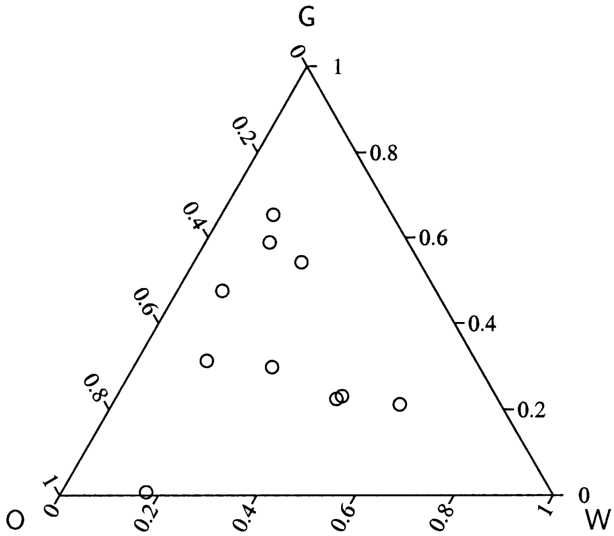

Figure 3. Ten saturation states from a uniform distribution used to define the initial and injected states of the Riemann problem.

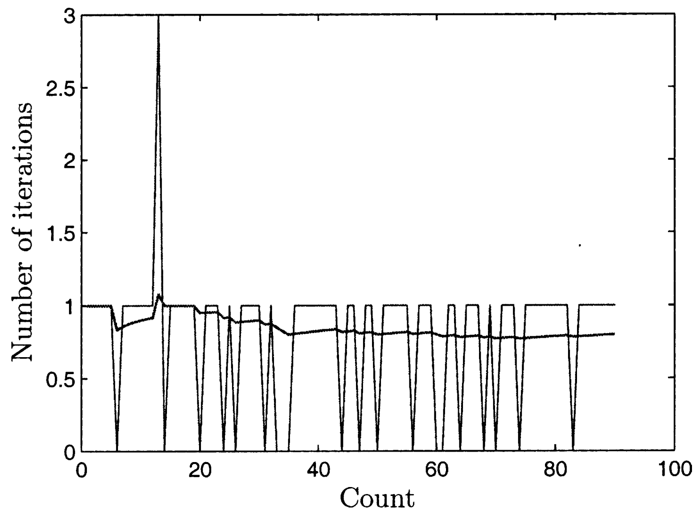

Figure 4. Number of iterations required for convergence of the algorithm in Figure 2 for all 90 combinations of initial and injected states. The thick solid line is the average number of iterations.

iterations are required. The proposed algorithm for finding the wave structure in three-phase flow is summarized in Figure 2.

2. Examples

Examples of all admissible wave configurations are shown in Juanes and Patzek (2004). Here, we use the same relative permeability functions and fluid viscosities, and study the performance of the algorithm to determine the wave structure. We take ten random saturation states from a uniform distribution on the ternary diagram (see Figure 3), and consider all possible combinations of initial and injected saturations. In Figure 4, we plot the number of iterations required for convergence for each pair of initial and injected states. The average number of iterations is about 0.8, demonstrating the effectiveness of the algorithm.

Reference

Juanes, R. and Patzek, T. W.: 2004, Analytical solution to the Riemann problem of three-phase flow in porous media. *Transp. Porous Media* **55**(1), 47–70, 2004.