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On the use of homotopy analysis method for solving unsteady MHD flow of Maxwellian fluids above impulsively stretching sheets

Amir Alizadeh-Pahlavan, Kayvan Sadeghy *

University of Tehran, Department of Mechanical Engineering, P.O. Box 11155-4563, Tehran, Iran

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ABSTRACT

In the present work, unsteady MHD flow of a Maxwellian fluid above an impulsively stretched sheet is studied under the assumption that boundary layer approximation is applicable. The objective is to find an analytical solution which can be used to check the performance of computational codes in cases where such an analytical solution does not exist. A convenient similarity transformation has been found to reduce the equations into a single highly nonlinear PDE. Homotopy analysis method (HAM) will be used to find an explicit analytical solution for the PDE so obtained. The effects of magnetic parameter, elasticity number, and the time elapsed are studied on the flow characteristics.

1. Introduction

Many fluids of industrial importance (e.g., polymeric liquids) are non-Newtonian. That is, they might exhibit dramatic deviation from Newtonian behavior depending on the flow configuration and/or the rate of deformation [1–4]. These fluids often obey nonlinear constitutive equations, and the complexity of their constitutive equation is the main culprit for the lack of exact analytical solution in all but a few simple cases. Unfortunately, the situation does not improve when one resorts to approximate theories such as creeping flow theory or boundary layer theory. That is, even the reduced set of governing equations obtained using such approximate theories are often found to be too formidable to render themselves to an analytical solution. Thus, it should not be surprising why the field of computational rheology is so active in relation to non-Newtonian fluids. But computer codes need to be verified, and this can only be done by confronting their output with an analytical solution.

Exact analytical solutions have indeed been found for some simple rheological models such as power-law model and/or second- and third-order models. But, for more realistic constitutive equations such as Maxwell model, Giesekus model, and/or PTT model, the likelihood of finding an exact solution is remote [5–9], and this is particularly so for unsteady flows. In the present work, we will try to find an exact analytical solution using homotopy analysis method for unsteady MHD flow of an upper-convected Maxwell (UCM) model above an impulsively stretched sheet assuming that the boundary layer approximation is applicable. MHD flows of non-Newtonian fluids in different geometries has been the subject of much study in the past [10–15]. But to the best of our knowledge, no analytical solution has previously been reported for unsteady MHD flow of viscoelastic fluids above a stretching sheet.

* Corresponding author. Tel.: +98 21 61114011; fax: +98 21 88013029.
E-mail address: sadeghy@ut.ac.ir (K. Sadeghy).

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2. Theoretical formulations

The first step in the boundary layer analysis of any viscoelastic fluid is to derive the boundary layer equations pertinent to that particular fluid. This can be achieved starting from the Cauchy equations of motion, which must include a source term due to the imposed magnetic field. For laminar two-dimensional flow induced in an otherwise quiescent, incompressible, electrically conducting UCM fluid resting stress-free above an impulsively stretched sheet it is easy to show that the x-momentum equation reduces simply to

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\beta\left[u^2\left(\frac{\partial^2 u}{\partial x^2}\right)+v^2\left(\frac{\partial^2 u}{\partial y^2}\right)+2uv\left(\frac{\partial^2 u}{\partial x\partial y}\right)\right]=\upsilon\left(\frac{\partial^2 u}{\partial y^2}\right)-\frac{\sigma B_0^2}{\rho}u \tag{1}$$

where B_0 is the strength of the magnetic field, υ is the kinematic viscosity of the fluid, and β is the relaxation time of the fluid. As to the boundary conditions, we are going to assume that the sheet is being stretched linearly. Therefore, the initial and boundary conditions become

$$t<0:\ @\ y=0;\ u=0, v=0;\ @\ y\to\infty;\ u\to 0 \tag{2}$$
$$t\geqslant 0:\ @\ y=0;\ u=ax, v=0;\ @\ y\to\infty;\ u\to 0 \tag{3}$$

in which a is a known constant. Eq. (1) combined with the continuity equation for an incompressible fluid, i.e., $\partial u/\partial x + \partial u/\partial y = 0$, constitute the two equations governing the transfer of momentum between an impulsively stretching sheet and the fluid surrounding it. Now, since the flow is two-dimensional, the concept of the stream function can be made use of , ψ, to combine the momentum and continuity equations into a single partial differential equation in terms of ψ. The equation so obtained can be transformed into an equivalent PDE using the technique of the similarity solution. To achieve this goal, we introduce the following similarity variables ξ, η:

$$\eta=y\sqrt{\frac{a}{\upsilon\xi}};\quad \xi=1-\exp(-\tau);\quad \tau=at \tag{4}$$

where τ is dimensionless time. The stream function can then be made dimensionless as

$$\psi=x\sqrt{a\upsilon\xi}f(\xi,\eta) \tag{5}$$

where $f(\xi,\eta)$ is the dimensionless stream function. The velocity components are related to the stream function as

$$u=\frac{\partial\psi}{\partial y};\quad v=-\frac{\partial\psi}{\partial x} \tag{6}$$

Substituting ψ into Eq. (1), the single PDE equation governing the flow is obtained as

$$f'''+\xi\left[ff''-(f')^2-Mf'\right]+K\xi\left[2ff'f''-f^2f'''\right]+\frac{1}{2}\eta(1-\xi)f''-\xi(1-\xi)\frac{\partial f'}{\partial\xi}=0 \tag{7}$$

where prime denotes differentiation with respect to η. The boundary conditions for this equation are

$$\begin{aligned}&f(\xi,0)=0;\quad f'(\xi,0)=1\\&\lim_{\eta\to\infty}f'(\xi,\eta)=0\end{aligned} \tag{8}$$

In Eq. (7), M and K are dimensionless numbers referred to as magnetic and elasticity numbers and defined as $M=\frac{\sigma B_0^2}{\rho\cdot a}$ and $K=\beta\cdot a$. It is to be noted that for the case of $\xi=0$ (which corresponds to $\tau=0$), Eq. (7) reduces, as it should, to

$$f'''+\frac{\eta}{2}f''=0 \tag{9}$$

This equation shows that at the beginning of the impulsive motion, neither the external magnetic field nor the fluid's elasticity has any effect on the flow field. Eq. (9) is known to have an analytical solution in terms of the error function; that is

$$f(0,\eta)=\eta-\eta\cdot\operatorname{Erfc}\left(\frac{\eta}{2}\right)+\frac{2}{\sqrt{\pi}}\left[1-\exp\left(-\frac{\eta^2}{4}\right)\right] \tag{10}$$

On the other hand, when $\xi=1$ (corresponding to $\tau\to\infty$), Eq. (7) reduces to

$$f'''+ff''-(f')^2-Mf'+K\left[2ff'f''-f^2f'''\right]=0 \tag{11}$$

This equation corresponds to the steady state case which has recently been solved by Alizadeh-Pahlavan et al. [15] using homotopy analysis method. For the case of $K=0$, this equation corresponds to the MHD flow of Newtonian fluid with an exact solution of

$$f(1,\eta)=\frac{1-\exp\left(-\sqrt{1+M}\eta\right)}{\sqrt{1+M}} \tag{12}$$

It is interesting to note that although for all values of ξ, $f'(\xi,\infty)$ tends to zero exponentially, but $f'(0,\infty)$ for the initial solution (see Eq. (10)) tends to zero much more quickly than $f'(1,\infty)$ of the steady solution (see Eq. (12)). So, mathematically, the initial solution (10) is different in essence from the steady state one. This can give us an idea of how hard it is to find an analytic solution uniformly valid at all times in the range of $0 \leqslant \tau < \infty$. Despite the difficulties inherent in Eq. (7), in the next section we will present an explicit analytic solution for this equation valid over the whole spatial and temporal domains.

3. Method of solution

In general, it is quite difficult to solve highly nonlinear partial differential equations analytically. The much celebrated perturbation technique can be used for this purpose but only for weakly nonlinear problems. In 1992, Liao [16,17] developed a new analytical technique called the homotopy analysis method (HAM) to tackle such nonlinear problems [18–21]. Being different from perturbation technique, HAM does not need any small parameter to function. As a matter of fact, the homotopy analysis method can be regarded as a unification of previous non-perturbation techniques such as Adomian method. By its very nature, HAM provides a family of series solutions whose convergence region can be adjusted and controlled by an auxiliary parameter. It is worth mentioning that the homotopy analysis method has successfully been applied to many nonlinear problems in solid and fluid mechanics [22–32]. Having said this, it should be conceded that the number of unsteady nonlinear problems solved using this method is rather limited [25–29].

The first step in the HAM is to find a set of base functions to express the sought solution of the problem under investigation. As mentioned by Liao [21], a solution may be expressed with different base functions, among which some converge to the exact solution of the problem faster than others. Such base functions are obviously better suited for the final solution to be expressed in terms of. Noting these facts, we have decided to express $f(\xi,\eta)$ by a set of base functions of the form

$$f(\xi,\eta) = \sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{i,j}^{k,m}\eta^j\xi^k \exp(-\mathrm{i}\eta) \tag{13}$$

where $a_{i,j}^{k,m}$ are the coefficients to be determined later using the governing equation, Eq. (7), and its corresponding boundary conditions (Eq. (8)). This expression (13) satisfies the rule of solution expression for $f(\xi,\eta)$ as required in the HAM [15,16]. According to this rule, and with an eye on the boundary conditions as given by Eq. (8), we choose an initial guess for the solution in the following form:

$$f_0(\xi,\eta) = 1 - \exp(-\eta) \tag{14}$$

In the second step, we need to choose an appropriate auxiliary linear operator. The basic idea of the HAM is to transform the nonlinear governing equation into an infinite number of linear equations with the advantage that unlike other techniques, one has the freedom to choose the linear operator. Moreover, in the HAM it is not necessary for the linear operator to be of the same order as the original nonlinear equation (for more details see Ref. [32]). Using these ideas, we introduce an auxiliary linear operator in the form of

$$L(f) = f''' + \lambda f'' \tag{15}$$

It is easy to check that this operator satisfies the following equation:

$$L\left[\frac{C_1 \mathrm{e}^{-\lambda\eta}}{\lambda^2} + C_2\eta + C_3\right] = 0 \tag{16}$$

where C_1, C_2, C_3 are the arbitrary constants. Based on Eq. (7), we are led to define the following nonlinear operator:

$$\Pi[F(\xi,\eta;q)] = F''' + \xi\left[FF'' - (F')^2 - MF'\right] + K\xi\left[2FF'F'' - F^2F'''\right] + \frac{1}{2}\eta(1-\xi)F'' - \xi(1-\xi)\frac{\partial F'}{\partial \xi} \tag{17}$$

where prime denotes differentiation with respect to η, and $F(\xi,\eta;q)$ is a kind of mapping function for $f(\xi,\eta)$ with q serving as an embedding parameter varying in the range of $[0,1]$. Using these operators, we can construct the so-called zero-order deformation equation as

$$(1-q)L[F(\xi,\eta;q) - f_0(\eta)] = q\hbar H(\xi,\eta)\Pi[F(\xi,\eta;q)] \tag{18}$$

where $\hbar$ is an auxiliary parameter, and $H(\xi,\eta)$ is an auxiliary function. The boundary conditions for this equation are

$$F(\xi,0;q) = 0, \quad F'(\xi,0;q) = 1, \quad F'(\xi,\infty;q) = 0 \tag{19}$$

Obviously, when $q = 0$ and $q = 1$, the above zeroth-order deformation equations (Eqs. (18) and (19)) have the following solutions:

$$F(\xi,\eta;0) = f_0(\xi,\eta) \tag{20}$$

and

$$F(\xi,\eta;1) = f(\xi,\eta) \tag{21}$$

Thus, as q increases from 0 to 1, the mapping function $F(\xi,\eta;q)$ varies from the initial guess, $f_0(\xi,\eta)$, to the final (still unknown) solution $f(\xi,\eta)$. Now expanding $F(\xi,\eta;q)$ by its Taylor series in terms of q, one would obtain

$$F(\xi,\eta;q) = f_0(\xi,\eta) + \sum_{m=1}^{\infty} f_m(\xi,\eta) q^m \tag{22}$$

where

$$f_m(\xi,\eta) = \frac{1}{m!}\frac{\partial^m F(\xi,\eta;q)}{\partial q^m}\bigg|_{q=0} \tag{23}$$

By setting $q = 1$ (assuming that the auxiliary linear operator L, the auxiliary function $H(\eta)$, and the auxiliary parameter $\hbar$ are all chosen appropriately) the final series solution becomes

$$f(\xi,\eta) = f_0(\xi,\eta) + \sum_{m=1}^{\infty} f_m(\xi,\eta) \tag{24}$$

To make the mathematics more tractable, we define the vector

$$\vec{f}_m = \{f_0, f_1, \ldots, f_m\} \tag{25}$$

At this stage in the analysis $f_m(\xi,\eta)$ is still unknown. Now, in order to obtain $f_m(\xi,\eta)$, we first differentiate the zero-order deformation equations m times with respect to q, divide the equation so obtained by $m!$ and finally set $q = 0$. With these maneuvering, the so-called high-order deformation equation is obtained for $f_m(\xi,\eta)$ as

$$L[f_m(\xi,\eta) - \chi_m f_{m-1}(\xi,\eta)] = \hbar H(\xi,\eta)\mathfrak{R}_m(\vec{f}_{m-1}) \tag{26}$$

where we have

$$\mathfrak{R}_m(\vec{f}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\Pi[F(\xi,\eta;q)]}{\partial q^{m-1}}\bigg|_{q=0} \tag{27}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1 \\ 1, & m > 1 \end{cases} \tag{28}$$

It should be noted that $\mathfrak{R}_m(\vec{f}_{m-1})$ only depends on $\vec{f}_{m-1}$ and is defined as

$$\mathfrak{R}_m(\vec{f}_{m-1}) = f'''_{m-1} - \xi\cdot M\cdot f'_{m-1} + \xi\cdot\sum_{k=0}^{m-1}\left[f_{m-1-k}f''_k - f'_{m-1-k}f'_k + Kf_{m-1-k}\sum_{l=0}^{k}\{2f'_{k-l}f''_l - f_{k-l}f'''_l\}\right] + (1-\xi)\left(\frac{\eta}{2}f''_{m-1} - \xi\frac{\partial f'_{m-1}}{\partial\xi}\right) \tag{29}$$

The boundary conditions for Eq. (26) are

$$\begin{cases} f_m(\xi,0) = 0 \\ f'_m(\xi,0) = 0 \\ f'_m(\xi,\infty) = 0 \end{cases} \tag{30}$$

Let $f^*_m(\xi,\eta)$ denote a special solution for Eq. (26). Based on Eq. (16), the general solution then becomes

$$f_m(\xi,\eta) = f^*_m(\xi,\eta) + C^m_1 e^{-\eta} + C^m_2\eta + C^m_3 + \chi_m f_{m-1}(\xi,\eta) \tag{31}$$

where the constants C^m_1, C^m_2, C^m_3 are determined by boundary conditions (Eq. (30)). In this way, it is easy to obtain the terms $f_m(\xi,\eta)$ of Eq. (26) one after the other (say, using symbolic softwares such as Mathematica or Maple). In order to satisfy the rule of solution expression, the rule of coefficient ergodicity, and the rule of solution existence [15,16], we have some freedom in choosing the auxiliary function $H(\xi,\eta)$. It is found that converged solutions can be obtained using either $H(\xi,\eta) = 1$or $H(\xi,\eta) = \exp(-\eta)$. In practice, however, it was realized that using the latter results in a series which converges faster. Therefore, in the following sections we use $H(\xi,\eta) = \exp(-\eta)$.

Now, in order to determine $f_m(\xi,\eta)$ one after the other we present the recursive formulae which give the solution of Eq. (7) in an explicit totally analytic form. Although it is quite challenging to obtain such recursive formulae for unsteady flows, it is worth as it can give us a better insight about the structure of the nonlinear problem at hand. To that end, it was found that the terms $f_m(\xi,\eta)$ can be expressed as

$$f_m(\xi,\eta) = \sum_{i=0}^{3m+1}\sum_{j=0}^{m}\sum_{k=0}^{m} a^{k,m}_{i,j} e^{-i\eta}\eta^j\xi^k \tag{32}$$

Substituting Eq. (32) in the high-order deformation Eq. (26), we obtain the coefficients $a_{i,j}^{k,m}$ for $m \geqslant 1$ as

$$\begin{aligned}
a_{0,0}^{k,m} &= \chi_m a_{0,0}^{k,m-1} + C_3^m \\
a_{0,j}^{k,m} &= \chi_m a_{0,j}^{k,m-1} \\
a_{1,0}^{k,m} &= \chi_m a_{1,0}^{k,m-1} + \sum_{r=0}^{m-1} \hbar \Delta_{0,r}^{k,m-1} \mu_{1,0}^{r} + C_1^m \\
a_{1,j}^{k,m} &= \chi_m a_{1,j}^{k,m-1} + \sum_{r=j-1}^{m-1} \hbar \Delta_{0,r}^{k,m-1} \mu_{1,j}^{r} \\
a_{i,0}^{k,m} &= \chi_m a_{i,0}^{k,m-1} - \sum_{r=0}^{m-1} \hbar \Delta_{i-1,r}^{k,m-1} \mu_{i,0}^{r} \\
a_{i,j}^{k,m} &= \chi_m a_{i,j}^{k,m-1} - \sum_{r=j-1}^{m-1} \hbar \Delta_{i-1,r}^{k,m-1} \mu_{i,j}^{r}
\end{aligned} \tag{33}$$

in which

$$\begin{aligned}
C_1^m &= \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} \hbar \Delta_{0,j}^{k,m-1} (\mu_{1,1}^{j} - \mu_{1,0}^{j}) \xi^k + \sum_{i=2}^{3m} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} \hbar \Delta_{i-1,j}^{k,m-1} (i\mu_{i,0}^{j} - \mu_{i,1}^{j}) \xi^k \\
C_2^m &= 0 \\
C_3^m &= \sum_{i=2}^{3m} \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} \hbar \Delta_{i-1,j}^{k,m-1} \mu_{i,0}^{j} \xi^k - \sum_{j=0}^{m-1} \sum_{k=0}^{m-1} \hbar \Delta_{0,j}^{k,m-1} \mu_{1,0}^{j} \xi^k - C_1^m
\end{aligned} \tag{34}$$

The definitions of $\Delta_{i,j}^{k,m}$ and $\mu_{n,k}^{q}$ in Eqs. (33) and (34) are

$$\begin{aligned}
\Delta_{i,j}^{k,m} &= d_{i,j}^{k,m-1} - M \times b_{i,j}^{k,m-1} + \Gamma_{i,j}^{k-1,m-1} + K \times \Lambda_{i,j}^{k-1,m-1} - \omega_{i,j}^{k-1,m-1} + \omega_{i,j}^{k-2,m-1} + \frac{1}{2} c_{i,j-1}^{k,m-1} - \frac{1}{2} c_{i,j-1}^{k-1,m-1} \\
\mu_{n,k}^{q} &= \begin{cases} \frac{q!}{k!}(q-k+2) & 0 \leqslant q, \quad 0 \leqslant k \leqslant q+1, \ n = 1 \\ \frac{q!}{k!} \sum_{p=0}^{q-k} \frac{(q-k-p+1)}{n^{q-k-p+2}(n-1)^{p+1}} & 0 \leqslant q, \ 0 \leqslant k \leqslant q, \ n \geqslant 2 \end{cases}
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
\Lambda_{i,j}^{k,m} = \sum_{s=0}^{m-1} \sum_{t=0}^{s} \sum_{i_1=\max\{0,i-3(m-s)+2\}}^{\min\{i,3s+2\}} \sum_{j_1=\max\{0,j-m+1+s\}}^{\min\{j,s\}} \sum_{k_1=\max\{0,k-m+s+1\}}^{\min\{k,s\}} \\
\sum_{n=\max\{0,i_1-3(s-t)-1\}}^{\min\{i_1,3t+1\}} \sum_{q=\max\{0,j_1-s+t\}}^{\min\{j_1,t\}} \sum_{l=\max\{0,k_1-s+t\}}^{\min\{k_1,t\}} a_{i-i_1,j-j_1}^{k-k_1,m-1-s} (2b_{i_1-n,j_1-q}^{k_1-l,s-t} \times c_{n,q}^{l,t} - a_{i_1-n,j_1-q}^{k_1-l,s-t} \times d_{n,q}^{l,t})
\end{aligned} \tag{36}$$

$$\Gamma_{i,j}^{k,m} = \sum_{s=0}^{m-1} \sum_{n=\max\{0,i-3(m-s)+2\}}^{\min\{3s+1,i\}} \sum_{q=\max\{0,j-m+s+1\}}^{\min\{j,s\}} \sum_{l=\max\{0,k-m+s+1\}}^{\min\{k,s\}} (a_{i-n,j-q}^{k-l,m-1-s} c_{n,q}^{l,s} - b_{i-n,j-q}^{k-l,m-1-s} b_{n,q}^{l,s}) \tag{37}$$

and

$$\begin{aligned}
\omega_{i,j}^{k,m} &= (k+1) b_{i,j}^{k+1,m} \\
d_{i,j}^{k,m} &= (j+1) c_{i,j+1}^{k,m} - i c_{i,j}^{k,m} \\
c_{i,j}^{k,m} &= (j+1) b_{i,j+1}^{k,m} - i b_{i,j}^{k,m} \\
b_{i,j}^{k,m} &= (j+1) a_{i,j+1}^{k,m} - i a_{i,j}^{k,m}
\end{aligned} \tag{38}$$

Therefore, all the coefficients in the series solution can be explicitly determined knowing only the first two coefficients from the initial guess (Eq. (14)); that is

$$\begin{aligned}
a_{0,0}^{0,0} &= 1 \\
a_{1,0}^{0,0} &= -1
\end{aligned} \tag{39}$$

And finally the Mth order explicit totally analytic solution of Eq. (7) can be expressed as

$$f(\xi, \eta) = \sum_{m=0}^{M} \sum_{i=0}^{3m+1} \sum_{j=0}^{m} \sum_{k=0}^{m} a_{i,j}^{k,m} e^{-i\eta} \eta^j \xi^k \tag{40}$$

The above expression provides us with an analytical solution valid over the whole spatial and temporal domains. To the best of the authors' knowledge, such a kind of solution has never been reported in the literature for similar problems.

4. Results and discussion

As proved by Liao [21], as long as the series solution (Eq. (40)) is convergent, it should converge to one of the solutions of Eq. (7). Note that the Eq. (40) contains the auxiliary parameter $\hbar$ which is not yet defined. This parameter plays an important role in the framework of HAM. In fact, this parameter controls the rate of convergence and the convergence region of the series. Proper values for this auxiliary parameter can be found by plotting the so-called $\hbar$-curves. In the present work, it is found that by setting $\hbar = -8/10$, the series (Eq. (40)) converges rapidly in the whole temporal and spatial domain. In the present work, we have stopped at 10th order of approximation as it was found to yield very accurate results for the problem under consideration.

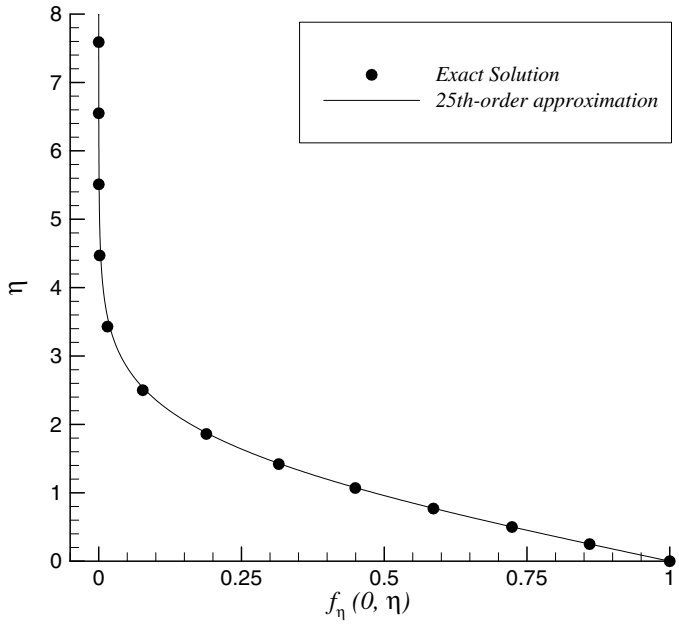

Fig. 1. Variation of $f_\eta(\xi,\eta)$ with η when $\xi = 0$. The solid line shows the 25th order HAM solution, and the symbols show the exact solution (Eq. (10)).

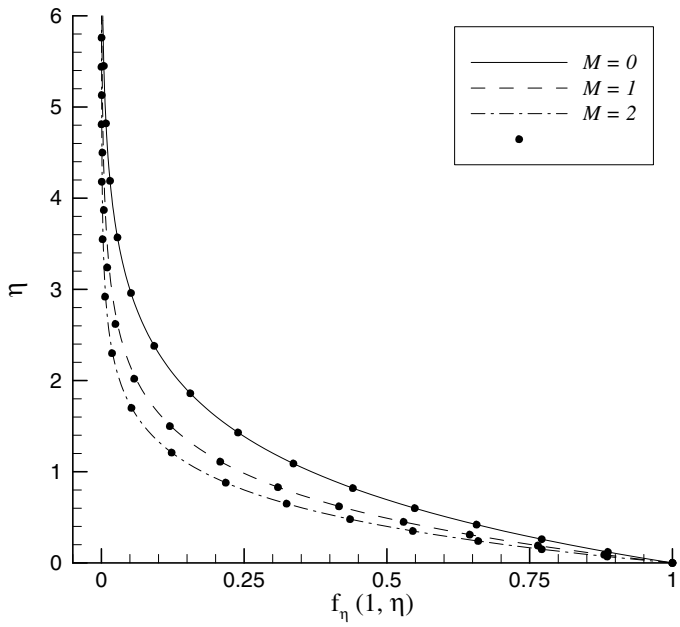

Fig. 2. Variation of $f_\eta(\xi,\eta)$ with η when $K = 0$, $\xi = 1$. The lines show 10th order HAM solutions and symbols show the exact solution (Eq. (12)).

As mentioned earlier, when $\xi = 0$, Eq. (7) has an exact solution in the form of Eq. (10). This exact solution helps us to verify our analytical solution. Fig. 1 shows our solution and the exact solution for the variation of $f_\eta(\xi,\eta)$ with η (for $\xi = 0$). Note that $f_\eta(\xi,\eta)$ indicates the non-dimensional velocity u/ax, and therefore shows the dimensionless velocity profiles. As can be seen in this figure, our solution compares well with the exact solution in the whole spatial domain when $\xi = 0$. As another verification of the HAM scheme described above, we have compared our results with the exact solution given by Eq. (12) (this exact solution is for steady flows of Newtonian fluids above a stretching sheet, i.e. $K = 0$, $\xi = 1$). Fig. 2 shows that our solution is virtually the same as the exact solution in the whole spatial domain when $K = 0$, $\xi = 1$. This figure also shows that by an increase in the magnetic parameter, the boundary layer thickness is decreased.

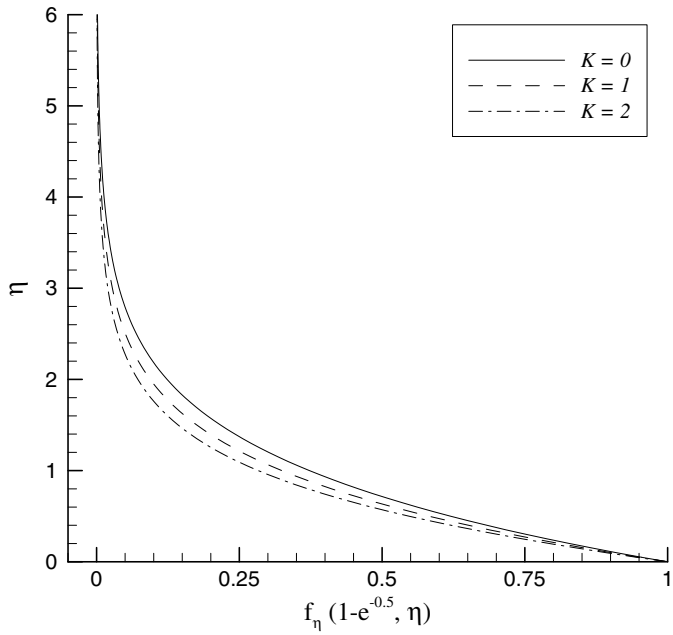

Fig. 3. The effect of the elasticity parameter, K, on the dimensionless velocity profile $f_\eta(\xi,\eta)$ for $\tau = 0.5$ and $M = 1$.

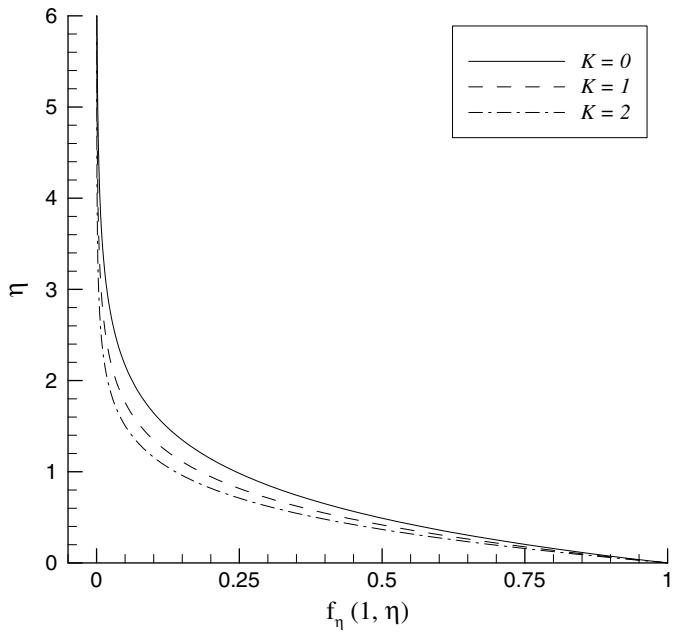

Fig. 4. The effect of elasticity parameter, K, on the steady dimensionless velocity profile $f_\eta(\xi,\eta)$ for $\xi = 1$ and $M = 1$.

Having validated the code developed in this work, we are now at a stage to present our results for unsteady MHD flow of a UCM fluid above an impulsively stretched sheet. We are going to investigate the effects of magnetic parameter, elasticity parameter, and time on the dimensionless velocity f_η, and also on the $f_{\eta\eta}$ (which is related to the wall shear stress). Fig. 3 shows the effect of K on the dimensionless velocity profiles $f_\eta(\xi,\eta)$ for the case of $\tau = 0.5$ and $M = 1$. This figure shows that by an increase in the elasticity parameter, a decrease in the boundary layer thickness should be expected. Fig. 4 shows the effect of K on dimensionless velocity profile $f_\eta(\xi,\eta)$ when $\xi = 1$ and $M = 1$. This figure also includes the steady state flow filed as recently obtained by Alizadeh-Pahlavan et al. [15]. This figure again shows that an increase in the elasticity parameter results in a decrease in the boundary layer thickness. As previously mentioned, for the case of $\xi = 0$, the elasticity

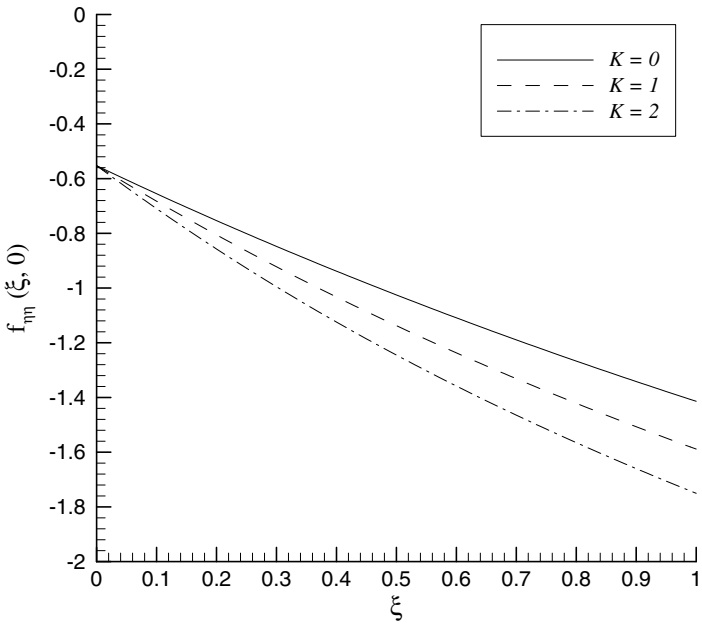

Fig. 5. The effect of the elasticity parameter, K, on $f_{\eta\eta}(\xi,\eta)$ for $\eta = 0$ and $M = 1$.

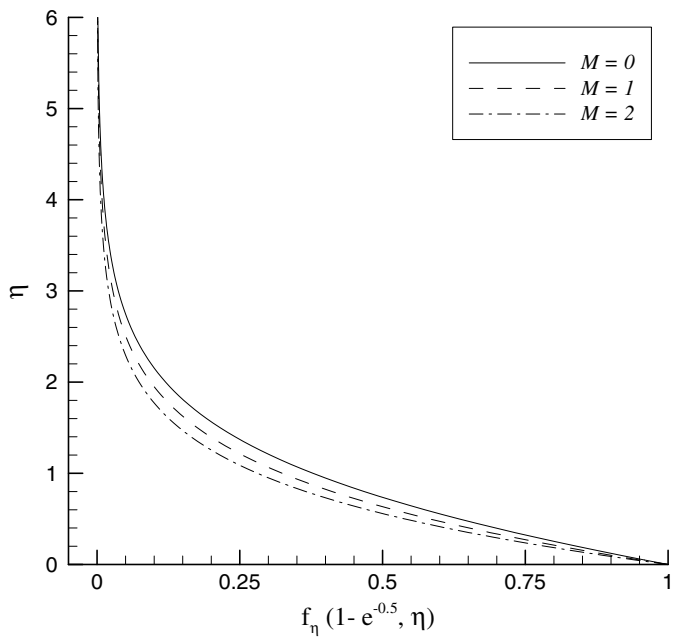

Fig. 6. The effect of magnetic parameter, M, on the dimensionless velocity profile $f_\eta(\xi,\eta)$ for $\tau = 0.5$ and $K = 1$.

parameter has no effect on the flow field. Thus, it can be concluded that as the flow reaches the steady state, the effect of the elasticity parameter becomes more pronounced. Fig. 5 shows the effect of the elasticity parameter on $f_{\eta\eta}(\xi,0)$ for $M = 1$ in the domain of $0 \leqslant \xi \leqslant 1$ corresponding to $0 \leqslant \tau < \infty$. This figure shows that by increasing K a decrease in $f_{\eta\eta}(\xi,0)$ can be expected. Again, the effect of elasticity parameter becomes more pronounced as the flow tends to the steady state. The effect of the elasticity parameter on $f_\eta(\xi,\eta)$ and $f_{\eta\eta}(\xi,0)$ for the steady flow (i.e., for $\xi = 1$) is seen to be virtually the same as those reported recently by Alizadeh-Pahlavan et al. [15].

Figs. 6 and 7 show the effect of the magnetic parameter on the dimensionless velocity profile $f_\eta(\xi,\eta)$ for $K = 1$ at $\tau = 0.5$ and $\xi = 1$. These figures show that by increasing the magnetic parameter a decrease in the boundary layer thickness will result.

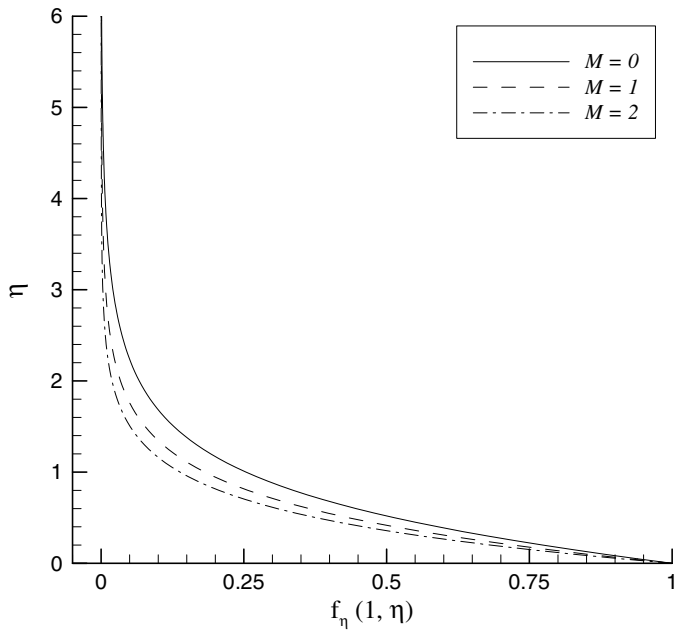

Fig. 7. The effect of magnetic parameter, M, on the steady dimensionless velocity profile $f_\eta(\xi,\eta)$ for $\xi = 1$ and $K = 1$.

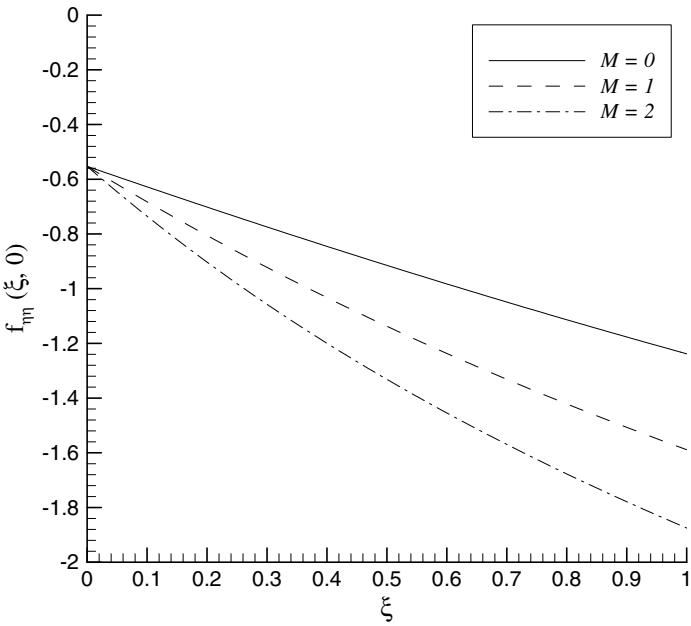

Fig. 8. The effect of magnetic parameter, M, on $f_{\eta\eta}(\xi,\eta)$ for $\eta = 0$ and $K = 1$.

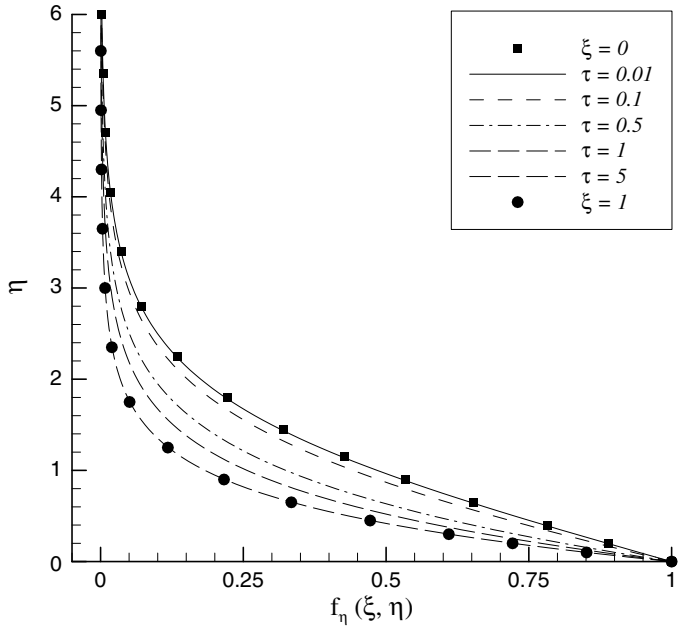

Fig. 9. The effect of dimensionless time, τ, on dimensionless velocity profile $f_\eta(\xi,\eta)$ for $K = 1$ and $M = 1$; the squares show the exact solution (Eq. (10)) and the circles show the steady solution.

And, this effect becomes more pronounced as the flow reaches the steady state. The effect of the magnetic parameter in Fig. 7 for the steady solution is the same as the results obtained by Alizadeh-Pahlavan et al. [15]. The effect of the magnetic parameter on $f_{\eta\eta}(\xi,0)$ is shown in Fig. 8 for $K = 1$. This figure shows that an increase in the magnetic parameter causes a decrease in the $f_{\eta\eta}(\xi,0)$. And again, as Figs. 6 and 7 suggest, the effect of the magnetic parameter becomes more noticeable as the flow tends to the steady state. The results presented in Fig. 8 compares well with those reported recently by Alizadeh-Pahlavan et al. [15] for the steady flow, i.e. for $\xi = 1$.

Fig. 9 shows the effect of dimensionless time, τ, on the velocity profiles $f_\eta(\xi,\eta)$ for $K = 1$ and $M = 1$. As seen in this figure, the flow reaches to the steady state roughly at $\tau = 5$. This means that the transition from the rest state to the steady state is a rapid process. It is to be noted that as shown in Fig. 9, the case of $\xi = 0$ and $\xi = 1$ correspond to $\tau = 0$ and $\tau \to \infty$, respectively. This figure shows that as the flow approaches the steady state, the thickness of the boundary layer is progressively decreased, as expected.

5. Concluding remarks

In the present work, we have obtained an analytical solution for the unsteady MHD flow of UCM fluid above an impulsively stretched sheet using homotopy analysis method (HAM). The solution is presented in an explicit totally analytic manner in recursive formulae. It is shown that our solution is valid for the whole temporal ($0 \leqslant \tau < \infty$) and spatial domain ($0 \leqslant \eta < \infty$. The effect of the elasticity parameter, the magnetic parameter, and the time elapsed since the motion started are investigated on the dimensionless velocity profile $f_\eta(\xi,\eta)$, and also on the $f_{\eta\eta}(\xi,0)$. It is concluded that by an increase in any of these parameters, the thickness of the boundary layer is decreased. Simultaneously, a drop in the wall shear stress, as reflected in $f_{\eta\eta}(\xi,0)$, can be expected by an increase in these parameters. These results are obviously of technological importance as they offer means to alter flow kinematics and/or to reduce the force required to pull the sheet. Alternatively, in cases where the flow is involved with significant heat transfer, the rate of cooling can be controlled by judicious choice of these parameters. Last but not least of all is the notion that the analytical approach drawn in this work is of a general nature meaning that it can easily be extended to other unsteady flow problems dealing with Newtonian and/or non-Newtonian fluids.

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