

On the analytical solution of viscous fluid flow past a flat plate

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Abstract

In this Letter, we propose a simple approach using HAM to obtain accurate totally analytical solution of viscous fluid flow over a flat plate. First, we show that the solution obtained using HPM is not a reliable one; moreover, we show that HPM is only a special case of HAM and its basic assumptions are restrictive rather than useful. We set $\hbar = -1$ for the case of comparison of our results to those obtained using HPM. Afterwards, we introduce an extra auxiliary parameter and a straightforward approach to find best values of this auxiliary parameter which plays a prominent role in the frame of our solution and makes it more convergent in comparison to previous works.

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1. Introduction

Ludwig Prandtl's first student, H. Blasius, found a celebrated solution for laminar boundary layer flow past a flat plate in 1908 [1]. His solution is of great historical importance in fluid mechanics, and is referred to as the first exact solution of Navier–Stokes equations. In fact, his solution is regarded as an exact one since he transformed the PDE equations of Navier–Stokes to a nonlinear ODE equation [2]. Although this equation is easy to solve using numerical algorithms, it had never yielded to exact analytic solution until recently. Blasius himself gave matching inner and outer series solutions [1]. His solution can be thought of as semi analytical and semi numerical one which can be obtained by perturbation methods; however, it is unfortunately only convergent in a rather restricted region ($|\eta| \leq \rho_0$) where $\rho_0 \approx 5.690$. Almost a century later, Liao [3] proposed an explicit totally analytic solution for Blasius equation using Homotopy analysis method (HAM) for the first time. HAM was proposed by Liao in his PHD dissertation [4] and was later

developed and improved in his book [5], “Beyond perturbation”. Many of previous non-perturbation techniques are just special cases of this method. HAM is also successfully applied to many nonlinear problems in science and engineering [6–17]. Later in 1999, He [18] introduced the so-called Homotopy perturbation method (HPM) and then in 2004 used HPM to solve Blasius equation and compared his results to those obtained by Liao [19]. However, as shown by Liao in [20], his approach includes some evident mathematical mistakes. Moreover, HPM is just a special case of HAM as shown by Liao and others [21–26]; therefore, it cannot provide anything new.

In this Letter, the basic ideas of HAM are briefly introduced and then previous works on analytical solution of Blasius equation are analyzed. Afterwards, we present our method of solution in which a new auxiliary parameter different from the conventional \hbar in HAM is introduced in order to ensure the convergence of the series solution. Finally, the obtained results are compared to those of He [19] and Liao [3].

2. Brief description of HAM

Let us consider a nonlinear differential equation in the form:

$$N[f(x)] = 0, \quad (1)$$

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where N is a nonlinear operator and $f(x)$ is an unknown function. The basic concept of HAM like many other previous techniques is to transform this nonlinear equation in to an infinite number of linear equations. But the main breakthrough in HAM is that unlike all previous techniques, it provides us with great freedom to choose the linear operator. This freedom leads to solution of many highly nonlinear equations which have not been solved analytically before (for more details refer to [21]).

Using an embedding parameter $q \in [0, 1]$, we construct a Homotopy:

$$(1 - q)L[F(x; q) - f_0(x)] - q\hbar N[F(x; q)] = 0, \tag{2}$$

where L is a linear operator with the property $L[0] = 0$, \hbar is a non-zero auxiliary parameter and $f_0(x)$ is an initial guess for the unknown function $f(x)$. Eq. (2) is called zeroth-order deformation equation. It is obvious that as q increases from 0 to 1, the above equation changes from a simple linear one to the original nonlinear equation; that is $F(x; q)$ deforms from an initial guess $f_0(x)$ to the exact solution $f(x)$ of the original equation.

$F(x; q)$ can be expressed using Taylor series in powers of q as follows:

$$F(x; q) = F(x; 0) + \sum_{n=1}^{\infty} \frac{q^n}{n!} \left. \frac{\partial^n F(x; q)}{\partial q^n} \right|_{q=0}. \tag{3}$$

Then writing

$$f_n(x) = \left. \frac{1}{n!} \frac{\partial^n F(x; q)}{\partial q^n} \right|_{q=0} \tag{4}$$

we will have:

$$F(x; q) = f_0(x) + \sum_{n=0}^{\infty} f_n(x)q^n. \tag{5}$$

Assuming that the linear operator L and the auxiliary parameter \hbar are properly chosen that the above series is convergent, and setting $q = 1$, the solution of Eq. (1) can be expressed as:

$$f(x) = f_0(x) + \sum_{n=0}^{\infty} f_n(x). \tag{6}$$

The terms $f_n(x)$ are unknown yet. There are two ways to obtain these terms as shown by Liao [21].

The first way which is used by most HAM users is to construct the so-called *high-order deformation equation*. Differentiating the *zeroth-order deformation equation* n times with respect to q and dividing by $n!$ and finally setting $q = 0$, we have the so-called *n th-order deformation equation*:

$$L[f_n(x) - \chi_n f_{n-1}(x)] = \hbar \Re_n[\vec{f}_{n-1}, x] \tag{7}$$

in which:

$$\vec{f}_{n-1} = \{f_0(x), f_1(x), \dots, f_{n-1}(x)\} \tag{8}$$

and

$$\Re_n[\vec{f}_{n-1}, x] = \left. \frac{1}{(n-1)!} \frac{\partial^{n-1} F(x; q)}{\partial q^{n-1}} \right|_{q=0}, \tag{9}$$

$$\chi_n = \begin{cases} 0 & \text{for } n \leq 1, \\ 1 & \text{for } n > 1. \end{cases} \tag{10}$$

The right-hand side of Eq. (7) is only dependent on \vec{f}_{n-1} ; therefore, it is easy to obtain the remaining terms of Eq. (6) which provides us with the solution of Eq. (1).

The second way to obtain the terms $f_n(x)$ is to substitute $F(x; q)$ as defined by Eq. (5), in the *zeroth-order deformation equation*; then, equating the same powers of q provides us with infinite number of linear equations which give the terms $f_n(x)$.

The two foregoing ways of obtaining the terms $f_n(x)$ are in fact the same according to fundamental rules of calculus [21].

Both the linear operator L and auxiliary parameter \hbar play important roles in the frame of HAM [5,21]. It is crucial to select a proper linear operator; however, it is not enough to ensure the convergence of solution. In fact, \hbar provides us with a simple way to control the convergence range and rate of series solution which is shown in Eq. (6). Regarding these facts about HAM, it is obvious that HPM proposed by He [18] is only a special case of HAM when $\hbar = -1$, therefore HPM is less efficient in ensuring the convergence of solution. Moreover, HPM's main assumption that the embedding parameter (q) is a small parameter not only is of no use, but also is restrictive as will be shown later.

In the next section, some of previous analytical solutions of Blasius equation will be analyzed and it will be proved that some of them are not mathematically reliable.

3. Analysis of previous analytical solutions of Blasius equation

Blasius used the boundary layer approximations introduced by Prandtl [2] together with the assumption of constant free stream velocity and therefore zero pressure gradient. Then, he introduced the following similarity transformation:

$$\begin{aligned} \psi(x, y) &= \sqrt{vUx} f(\eta), \\ \eta &= \frac{y}{\sqrt{vx/U}} \end{aligned} \tag{11}$$

in which $\psi(x, y)$ is stream function. Then, using these transformations, the Navier–Stokes equations reduce to:

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0. \tag{12a}$$

Subject to boundary conditions:

$$f(0) = f'(0) = 0, \tag{12b}$$

$$f'(\infty) = 1. \tag{12c}$$

The boundary conditions (12b) and (12c) follow from the no-slip boundary conditions and the matching condition with outer flow, respectively.

To solve the Blasius equation using HAM, Liao [3] introduced the following linear operator:

$$L[f] = f''' + \beta f'' \tag{13}$$

He presented his solution with recurrence formulae in the form:

$$f(\eta) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n} \eta^n \exp(-m\eta). \tag{14}$$

Later in 2004, He [19] used the so-called HPM to obtain the solution of Blasius equation. He used the same linear operator as Eq. (13). In order to find the proper values of β for his solution, he decided to avoid the secular terms $\eta^n e^{-\beta\eta}$. However, as mentioned by Liao in [20], the terms $\eta^n e^{-\beta\eta}$ are not secular at all since for $\beta \geq 0$:

$$\lim_{\eta \rightarrow \infty} \eta^n e^{-\beta\eta} = 0. \tag{15}$$

Therefore, unlike the terms such as $(t \cos t)$ that tend to infinity as $t \rightarrow \infty$, the terms $\eta^n e^{-\beta\eta}$ are not secular at all. Thus the approach introduced by He [19] is not a mathematically reliable one.

In the other part of his work, He [19] made use of another linear operator:

$$L[f] = f''' + 1 \times f'' \tag{16}$$

Then, he expanded the constant 1 into power series of q . This approach is again mathematically incorrect since it is not possible to expand a constant in Taylor series. However, it is possible to define a mapping like this:

$$1 = a_0 + a_1 q^1 + a_2 q^2 + \dots \tag{17}$$

For which, the right-hand side, deforms from an initial guess to its final value, while q deforms from 0 to 1. Definitely, it is not possible to define such kind of mapping in HPM, since q is assumed to be a small parameter. This indeed emphasizes the fact that assuming q to be a small parameter is restrictive rather than helpful.

In the following section we make use of a rather different approach from HAM to ensure the convergence of solution and compare our results to those of He [19] and Liao [3].

4. Method of solution

In the present work, although it is not reasonable to set $h = -1$, we do so in order to compare our approach to that of He [19]. Therefore, the *zeroth-order deformation equation* becomes:

$$(1 - q)L[F(x; q) - f_0(x)] + qN[F(x; q)] = 0. \tag{18}$$

We make use of the same operator as the one used by Liao [3]. Therefore, we have:

$$(1 - q)[f'''(\eta) + \beta f''(\eta)] + q \left[f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) \right] = 0. \tag{19}$$

Subject to the boundary conditions:

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1. \tag{20}$$

It is easy to notice that as q deforms from 0 to 1, Eq. (19) changes from the initial linear equation $f'''(\eta) + \beta f''(\eta) = 0$ to the Blasius equation.

Then putting Eq. (6) in (19) and equating the same powers of q , we have:

$$q^0: \quad f_0'''(\eta) + \beta f_0''(\eta) = 0, \\ f_0(0) = 0, \quad f_0'(0) = 0, \quad f_0'(\infty) = 1; \tag{21}$$

$$q^1: \quad f_1'''(\eta) + \beta f_1''(\eta) + \frac{1}{2} f_0(\eta) f_0''(\eta) - \beta f_0''(\eta) = 0; \tag{22}$$

$$q^2: \quad f_2'''(\eta) + \beta f_2''(\eta) + \frac{1}{2} f_0(\eta) f_1''(\eta) - \beta f_1''(\eta) \\ + \frac{1}{2} f_1(\eta) f_0''(\eta) = 0 \tag{23}$$

⋮

in which, for $q^n, n > 0$ we have:

$$f_n(0) = 0, \quad f_n'(0) = 0, \quad f_n'(\infty) = 0. \tag{24}$$

Up to now, our method is generally the same as that of He [19]; however, we do not treat q as a small parameter since it is not helpful at all. Moreover, unlike He we rely on another approach to find the proper values of β in order to make our series solution convergent. To do so, we follow a similar approach as what is used in HAM to find proper values of h . Let us consider a physical parameter of the problem, e.g. $f''(0)$, which corresponds to wall skin friction. We have:

$$f''(0) = \sum_{n=0}^{\infty} f_n''(0). \tag{25}$$

If the solution of original nonlinear equation is unique and if the series solution given in Eq. (25) converges, it should always converge to the same value. Both the series solutions for $f(\eta)$ and $f''(0)$ contain β . Therefore, we have a family of solutions in terms of β among which we may find the most convergent one using the best value of β . In fact, plotting $f''(0)$ versus β , there should exist a horizontal segment in this plot as long as the solution series given in Eq. (25) is convergent. Hence, we may find the proper values of β which correspond to this horizontal segment to ensure that our solution is convergent. In a recent work, Alizadeh-Pahlavan et al. [17] used a similar approach by introducing a two-auxiliary parameter HAM in order to solve a strongly nonlinear problem.

5. Results and discussion

Fig. 1 shows the variation of $f''(0)$ with respect to β for different orders of approximation; it is evident that increasing the order of approximation results in a wider horizontal segment in the plot which consequently results in a more accurate solution. To have a better insight to the structure of the solution, $f''(0)$ is presented here as a function of β for different orders of approximation:

2nd order:

$$f''(0) = -\frac{5}{24\beta^3} + \frac{1}{2\beta}; \tag{26}$$

4th order:

$$f''(0) = \frac{1}{\beta} - \frac{25}{12\beta^3} + \frac{275}{96\beta^5} - \frac{4879}{2880\beta^7}; \tag{27}$$

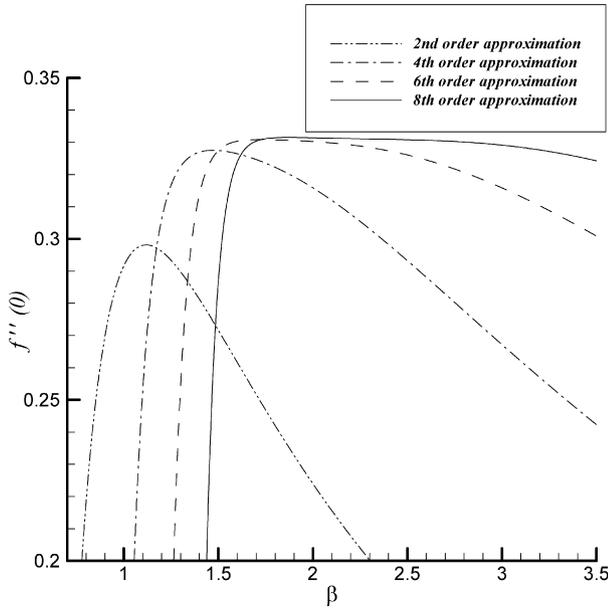


Fig. 1. Variation of $f''(0)$ with β for different orders of approximation.

Table 1
Comparison of our results to those of Liao [3] and also best values of β

Our results			Liao's results [3]	
Order of approximation (n)	Best value of β	$f''(0)$	Order of approximation (n)	$f''(0)$
2	1.10	0.298022	5	0.28098
4	1.47	0.327531	10	0.32992
6	1.72	0.330855	15	0.33164
8	1.87	0.331503	20	0.33198
10	2	0.331807	25	0.33204

6th order:

$$f''(0) = \frac{3}{2\beta} - \frac{175}{24\beta^3} + \frac{1925}{72\beta^5} - \frac{4879}{80\beta^7} + \frac{2740789}{34560\beta^9} - \frac{36712201}{806400\beta^{11}}; \tag{28}$$

8th order:

$$f''(0) = \frac{2}{\beta} - \frac{35}{2\beta^3} + \frac{1925}{16\beta^5} - \frac{53669}{96\beta^7} + \frac{30148679}{17280\beta^9} - \frac{477258613}{134400\beta^{11}} + \frac{2500661642369}{580608000\beta^{13}} - \frac{3488896893422083}{1463132160000\beta^{15}}. \tag{29}$$

The best values of β and their corresponding values of $f''(0)$ are provided in Table 1 for different orders of approximation. Howarth [27,28] obtained an accurate numerical solution for Blasius equation in which $f''(0) = 0.332057$. Our results are in good consistency with those of Howarth. Moreover, as shown in Table 1, our series solution converges to the accurate solution of Howarth faster than that of Liao [3]. We do not bring He's results [19] here, since his method of solution is not mathematically reliable. Fig. 2 shows the plot of Blasius functions f ,

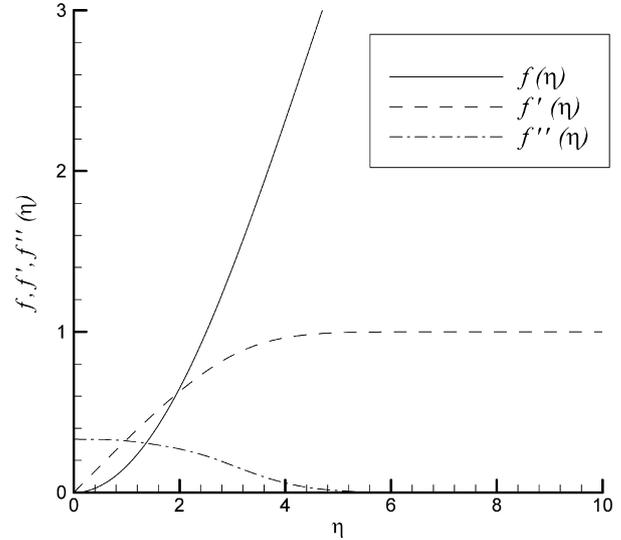


Fig. 2. Blasius functions for 10th order of approximation when $\beta = 2$.

f' and f'' which indeed emphasizes the fact that our solution is convergent in the whole domain $\eta \in [0, \infty]$ unlike the traditional perturbation techniques which were only convergent in the near vicinity of the origin $\eta = 0$.

6. Conclusion

In the present work, we made use of HAM to solve Blasius equation analytically. First, we discussed recent works by Liao [3] and He [19] that provided an analytical solution for this equation. Then, our approach was presented in which use was made of another auxiliary parameter other than the conventional \hbar in HAM to ensure the convergence of the series solution. Although \hbar plays a prominent role in the frame of HAM, we showed that it is possible to introduce other auxiliary parameters to ensure the convergence of solution and this indeed shows great flexibility of HAM to solve nonlinear ODEs and PDEs arising in science and engineering. Moreover, we showed that our series solution converges faster than that of Liao [3] due to using the best values of β . It should be noted that the approach used in the present work has a general meaning and therefore can be used widely when applying HAM to different nonlinear problems. This approach is especially well suited for the problems with strong nonlinearity in which the conventional auxiliary parameter \hbar in HAM is not enough to ensure the convergence of the series solution as was recently shown by Alizadeh-Pahlavan et al. [17].

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